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## 1 Review of CBN and CBV Semantics

$$\begin{array}{rcl} e & ::= & x \mid \lambda\, x\, e \mid e_0\, e_1 \\ v & ::= & \lambda\, x\, e \end{array}$$

Call By Name Semantics:

$$\begin{array}{rcl} C[(\lambda\, x\, e_0)\, e_1] & \mapsto & C[e_0\, \{e_1/x\}] \\ C & ::= & [\, .\, ] \mid C\, e \end{array}$$

Call By Value Semantics:

$$\begin{array}{rcl} C[(\lambda\, x\, e)\, v] & \mapsto & C[e\, \{v/x\}] \\ C & ::= & [\, .\, ] \mid C\, e \mid v\, C \end{array}$$

## 2 CBV Translation and Notes

Call By Name to Call By Value Translation For compactness we omit the name of the translation and just treat $[\![.]\!]$ as a semantic function itself

$$\begin{array}{rcl} [\![x]\!] & = & x\, I = x\, (\lambda\, y\, y) \\ [\![\lambda\, x\, e]\!] & = & \lambda\, x\, [\![e]\!] \\ [\![e_0\, e_1]\!] & = & [\![e_0]\!]\, (\lambda\, z\, [\![e_1]\!]) \end{array}$$

Expresing semantics through translations is a style of semantics known as *denotational semantics*, although the target language is usualy mathematical functions rather than $\lambda$ -calculus terms. We'll see true denotational semantics later in the course.

## 3 Soundness and Adequacy in CBV semantics

$$\begin{array}{l} Soundness:\ e \mapsto^* v \Rightarrow \exists v' [\![e]\!] \mapsto^*\ v' \wedge v' \approx [\![v]\!] \\ Adequacy:\ \exists v'\ e \mapsto^* v\ \wedge\ v' \approx [\![v]\!] \Leftarrow [\![e]\!] \mapsto^* v' \end{array}$$

Basic idea of soundness is saying that the operational semantics doesn't break the meaning (with respect to the translation) of the program as it executes.

**Proof of soundness**

We will show that if $e \mapsto e'$ in CBN then $[\![e]\!] \approx [\![e']\!]$
We will prove this by induction on the form of $C_N$.

For $C_N$=[ ] we have $C_N[(\lambda\ x\ e_0)\ e1] \mapsto C_N[e_0\{e_1/x\}]$ or equivalently $(\lambda\ x\ e_0)\ e_1 \mapsto e_0\{e_1/x\}$. So we have to show that $[\![e_0]\!]\{\lambda\ z\ [\![e_1]\!]/x\} \approx [\![e_0\{e_1/x\}]\!]$.

We will show this by structural induction on $e_0$.

If $e_0 = x$ then we have:

$$[\![e_0]\!]\{\lambda\ z\ [\![e_1]\!]/x\} \approx (x\ I)\{\lambda\ z\ [\![e_1]\!]/x\} \approx \lambda\ z\ [\![e_1]\!]\ I \approx [\![e_1]\!] \approx [\![x\{e_1/x\}]\!] \approx [\![e_0\{e_1/x\}]\!]$$

If $e_0 = y$ with $y \neq x$ then we have:

$$[\![e_0]\!]\{\lambda\ z\ [\![e_1]\!]/x\} \approx (y\ I)\{\lambda\ z\ [\![e_1]\!]/x\} \approx y\ I \approx [\![y]\!] \approx [\![y\{e_1/x\}]\!] \approx [\![e_0\{e_1/x\}]\!]$$

If $e_0 = \lambda\ x\ e_2$ we have:

$$[\![\lambda\ x\ e_2]\!]\{\lambda\ z\ [\![e_1]\!]/x\} \approx (\lambda\ x\ [\![e_2]\!])\{\lambda\ z\ [\![e_1]\!]/x\} \approx [\![(\lambda\ x\ e_2)\{e_1/x\}]\!] \approx [\![e_0\{e_1/x\}]\!]$$

If $e_0 = \lambda\ y\ e_2$ we have:

$$[\![\lambda\ y\ e_2]\!]\{\lambda\ z\ [\![e_1]\!]/x\} \approx (\lambda\ y\ [\![e_2]\!])\{\lambda\ z\ [\![e_1]\!]/x\}$$

Given that $e_2$ is a subexpression of $e_0$ we can apply the induction hypothesis obtaining:

$$\lambda\ y\ ([\![e_2]\!]\{\lambda\ z\ [\![e_1]\!]/x\}) \approx \lambda\ y\ [\![e_2\{e_1/x\}]\!] \approx [\![(\lambda\ y\ e_2)\{e_1/x\}]\!] \approx [\![e_0\{e_1/x\}]\!]$$

If $e_0 = e_2\ e_3$ then we have:

$$[\![e_2\ e_3]\!]\{\lambda\ z\ [\![e_1]\!]/x\} \approx ([\![e_2]\!]\lambda\ z\ [\![e_3]\!])\{\lambda\ z\ [\![e_1]\!]/x\} \approx ([\![e_2]\!]\{\lambda\ z\ [\![e_1]\!]/x\})((\lambda\ z\ [\![e_3]\!])\{\lambda\ z\ [\![e_1]\!]/x\}) \approx$$
$$\approx ([\![e_2]\!]\{\lambda\ z\ [\![e_1]\!]/x\})(\lambda\ z\ ([\![e_3]\!]\{\lambda\ z\ [\![e_1]\!]/x\}))$$

Given that $e_2$ and $e_3$ are subexpressions of $e_0$ we can apply the induction hypothesis obtaining:

$$([\![e_2]\!]\{\lambda\ z\ [\![e_1]\!]/x\})(\lambda\ z\ ([\![e_3]\!]\{\lambda\ z\ [\![e_1]\!]/x\})) \approx [\![e_2\{e_1/x\}]\!](\lambda\ z\ ([\![e_3\{e_1/x\}]\!])) \approx [\![e_2\{e_1/x\}\ e_3\{e_1/x\}]\!] \approx$$
$$\approx [\![(e_2\ e_3)\{e_1/x\}]\!] \approx [\![e_0\{e_1/x\}]\!]$$

This concludes our proof that $[\![e_0]\!]\{\lambda\ z\ [\![e_1]\!]/x\} \approx [\![e_0\{e_1/x\}]\!]$.

Now, for $C_N = C'_N\ e''$ we have $C'_N[(\lambda\ x\ e_0)\ e1]e'' \mapsto C'_N[e_0\{e_1/x\}]e''$. Because $C'_N$ is a subexpression of $C_N$ we have $[\![C'_N[(\lambda\ x\ e_0)\ e1]]\!] \approx [\![C'_N[e_0\{e_1/x\}]]\!]$ according to the induction hypothesis. So we have (with induction on structure of $C_N$,now):

$$[\![C'_N[(\lambda\ x\ e_0)\ e_1]e'']\!] \approx [\![C'_N[(\lambda\ x\ e_0)e_1]]\!](\lambda\ z\ [\![e'']\!]) \approx [\![C'_N[e_0\{e_1/x\}]]\!](\lambda\ z\ [\![e'']\!]) \approx [\![C'_N[e_0\{e_1/x\}]e'']\!]$$

# 4 Extending the CBV Lambda Calculus

## 4.1 Adding If's and booleans

$$\begin{array}{lcl} e & ::= & \ldots \mid \#\mathsf{t} \mid \#\mathsf{f} \mid \textbf{if}\, e_0\, \textbf{then}\, e_1\, \textbf{else}\, e_2 \\ v & ::= & \ldots \mid \#\mathsf{t} \mid \#\mathsf{f} \end{array}$$

SOS:

$$C \quad ::= \quad \ldots \mid \textbf{if}\, C\, \textbf{then}\, e_1\, \textbf{else}\, e_2$$

$$\frac{e \to e'}{\mathcal{C}[\![e]\!] \to \mathcal{C}[\![e']\!]} \qquad \overline{\textbf{if}\ \#\mathsf{t}\ \textbf{then}\ e_1\ \textbf{else}\ e_2 \to\ e_1} \qquad \overline{\textbf{if}\ \#\mathsf{f}\ \textbf{then}\ e_1\ \textbf{else}\ e_2 \to\ e_2}$$

$$\begin{array}{rcl} [\![\#\mathsf{t}]\!] & = & \lambda\ x\ \lambda\ y\ (x\ I) \\ [\![\#\mathsf{f}]\!] & = & \lambda\ x\ \lambda\ y\ (y\ I) \\ [\![\textbf{if}\, e_0\, \textbf{then}\, e_1\, \textbf{else}\, e_2]\!] & = & [\![e_0]\!]\ (\lambda\ z\ [\![e_1]\!])\ (\lambda\ z\ [\![e_2]\!]) \end{array}$$

Note that this translation has no error checking for the case where #t or #f are not first argument to an if expression.

### 4.2 Adding Let's

$$e \;::=\; \ldots \;|\; \textbf{let } x = e_1 \textbf{ in } e_2$$

SOS:

$$C \;::=\; \ldots \;|\; \textbf{let } x = C \textbf{ in } e_2$$

$$\frac{}{\textbf{let } x = v \textbf{ in } e \;\to e\{v/x\}}$$

$$[\![\textbf{let } x = e_1 \textbf{ in } e_2]\!] = (\lambda\ x\ [\![e_2]\!])\ [\![e_1]\!]$$

### 4.3 Adding Pairs

$$\begin{array}{rcl} e & ::= & \ldots \;|\; \langle e_1, e_2 \rangle \;|\; \textbf{left } e \;|\; \textbf{right } e \\ v & ::= & \ldots \;|\; \langle v_1, v_2 \rangle \end{array}$$

SOS:

$$C \;::=\; \ldots \;|\; \langle C, e \rangle \;|\; \langle v, C \rangle \;|\; \textbf{left } C \;|\; \textbf{right } C$$

$$\frac{}{\textbf{left } \langle v_1, v_2 \rangle \;\to v_1} \qquad \frac{}{\textbf{right } \langle v_1, v_2 \rangle \;\to v_2}$$

$$\begin{array}{rcl} [\![\langle e_1, e_2 \rangle]\!] & = & (\lambda\ x\ \lambda\ y\ \lambda\ f\ f\ x\ y)\ [\![e_1]\!]\ [\![e_2]\!] \\ [\![\textbf{left } e]\!] & = & [\![e]\!]\ (\lambda\ x\ \lambda\ y\ x) \\ [\![\textbf{right } e]\!] & = & [\![e]\!]\ (\lambda\ x\ \lambda\ y\ y) \end{array}$$