Write
\[ \alpha = \frac{1}{2}(x_{2n} + y_{2n}) \]

Then \( \alpha - x \geq \alpha - x_{2n} - |x_{2n} - x| \geq \frac{1}{2}(y_{2n} - x_{2n}) - (2n)^{-1} > 0 \)

Also, \( y - \alpha \geq y_{2n} - \alpha - |y_{2n} - y| \geq \frac{1}{2}(y_{2n} - x_{2n}) - (2n)^{-1} > 0 \)

Therefore \( x < \alpha < y \).

As a corollary, for each \( x \) in \( R \) and \( r \) in \( R^+ \) there exists \( \alpha \) in \( Q \) with \( |x - \alpha| < r \). Here is another corollary.

**Proposition 7** If \( x_1, \ldots, x_n \) are real numbers with \( x_1 + \cdots + x_n > 0 \), then \( x_i > 0 \) for some \( i \) (\( 1 \leq i \leq n \)).

**Proof** By Lemma 4 there exists a rational number \( \alpha \) with \( 0 < \alpha < x_1 + \cdots + x_n \). Let \( a_i \) (\( 1 \leq i \leq n \)) be a rational number with
\[ |x_i - a_i| < (2n)^{-1} \alpha \]

Then
\[ \sum_{i=1}^{n} a_i \geq \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} |x_i - a_i| > \frac{1}{2} \alpha \]

Therefore \( a_i > (2n)^{-1} \alpha \) for some \( i \). It follows that
\[ x_i \geq a_i - |x_i - a_i| > 0 \]

**Corollary** If \( x, y, \) and \( z \) are real numbers with \( y < z \), then either \( x < z \) or \( x > y \).

**Proof** Since \( x - x + x - y = z - y > 0 \), either \( x - x > 0 \) or \( x - y > 0 \), by Proposition 7.

The next lemma gives an extremely useful method for proving inequalities of the form \( x \leq y \).

**Lemma 5** Let \( x \) and \( y \) be real numbers such that the assumption \( x > y \) implies that \( 0 = 1 \). Then \( x \leq y \).

**Proof** Without loss of generality, we take \( y = 0 \). For each \( n \) in \( Z^+ \), either \( x_n \leq n^{-1} \) or \( x_n > n^{-1} \). The case \( x_n > n^{-1} \) is ruled out, since it implies \( x > 0 \). Therefore \( -x_n \geq -n^{-1} \), for all \( n \), so that \( -x \geq 0 \).

**Theorem 1** Let \( \{a_n\} \) be a sequence of real numbers. Let \( x \) and \( y \) be real numbers, \( x < y \). Then there exists a real number \( x \) with

\[ (2.22) \quad x_0 \leq x \leq y_0 \]

and

\[ (2.23) \quad x \neq a_n \quad (n \in Z^+) \]

**Proof** We construct by induction sequences \( \{x_n\} \) and \( \{y_n\} \) of rational numbers such that

\[ (2.24) \quad (i) \quad x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq y_n \leq y_0 \quad (m \geq n \geq 1) \]

\[ (ii) \quad x_n > a_n \text{ or } y_n < a_n \quad (n \geq 1) \]

\[ (iii) \quad y_n - x_n < n^{-1} \quad (n \geq 1) \]

Assume that \( n \geq 1 \) and that \( x_n, \ldots, x_{n-1}, y_n, \ldots, y_{n-1} \) have been constructed. Either \( a_n > x_{n-1} \) or \( a_n < y_{n-1} \). In case \( a_n > x_{n-1} \), let \( x_n \) be any rational number with \( x_{n-1} < x_n < \min \{a_n, y_{n-1}\} \), and let \( y_n \) be any rational number with \( x_n < y_n < \min \{a_n, y_{n-1}, x_n + n^{-1}\} \). Then the relevant inequalities are satisfied. In case \( a_n < y_{n-1} \), let \( y_n \) be any rational number with \( \max \{a_n, x_{n-1}, y_{n-1} - n^{-1}\} < y_n < y_{n-1} \), and let \( x_n \) be any rational number with \( \max \{a_n, x_{n-1}, y_{n-1} - n^{-1}\} < x_n < y_n \). Again, the relevant inequalities are satisfied. This completes the induction.

From (i) and (iii) it follows that
\[ |x_n - a_{n-1}| = x_n - x_{n-1} < y_n - x_n < n^{-1} \quad (m \geq n) \]

Similarly \( |y_n - y| < n^{-1} \) for \( m \geq n \). Therefore \( x = \{x_n\} \) and \( y = \{y_n\} \) are real numbers. By (ii), they are equal. By (i), \( x_n \leq x \) and \( y_n \geq y \) for all \( n \). If \( a_n < x_n \) then \( a_n < x \) so \( a_n \neq x \). If \( a_n > y_n \) then \( a_n > y = x \), so \( a_n \neq x \). Thus \( x \) satisfies (2.22) and (2.23).

Theorem 1 is the famous theorem of Cantor, that the real numbers are uncountable. The proof is essentially Cantor's "diagonal" proof. Both Cantor's theorem and his method of proof are of great importance.

The time has come to consider some counterexamples. Let \( \{a_n\} \) be a sequence of integers, each of which is \( 0 \) or \( 1 \), for which we are unable to prove either that \( n_0 = 1 \) for some \( k \) or that \( a_n = 0 \) for all \( k \). This corresponds to what Brouwer calls "a fugitive property of the natural numbers." Such a sequence can be defined, for example, as