For suppose that \( g(p) = i \). Then \( f(p, i) = o \), and for any \( j < i, f(p, j) = o \).

Since \( A \) represents \( f \) in \( Q \), we have

\[
\begin{align*}
\Gamma & A(p, i, o), \text{ and (if } i > o), \\
\Gamma & A(p, p, o), \\
\vdots & \\
\Gamma & A(p, i - 1, o).
\end{align*}
\]

\((o), ..., (i - 1)\), and 14.11 entail that

\[
\Gamma \forall w (w < i \rightarrow -A(p, w, o)),
\]

which, together with \((i)\), entails that \( \Gamma B(p, i). \)

We must show that \( \forall x_{n+1} (B(p, x_{n+1}) \rightarrow x_{n+1} = i) \). Assume \( B(p, x_{n+1}) \), i.e., \( A(p, x_{n+1}, o) \& \forall w (w < x_{n+1} \rightarrow -A(p, w, o)) \). From \((i)\) and

\[
\forall w (w < x_{n+1} \rightarrow -A(p, w, o)),
\]

we have \(-i < x_{n+1}. \)

From \( A(p, x_{n+1}, o) \) and \((i + 1)\), we have \(-x_{n+1} < i. \) Thus by 14.11 we have \( x_{n-1} = i. \) So \( \forall x_{n+1} (B(p, x_{n+1}) \rightarrow x_{n+1} = i) \).

**Exercises**

14.1 Verify the following assertion: all recursive functions are representable in the theory \( \langle R \rangle \) whose language is \( L \) and whose theorems are the consequences in \( L \) of the following infinitely many sentences:

\[ i + j \text{ for all } i, j \text{ such that } i + j; \]
\[ i + j = k \text{ for all } i, j, k \text{ such that } i + j = k; \]
\[ i \cdot j = k \text{ for all } i, j, k \text{ such that } i \cdot j = k; \]
\[ \forall x (x < i \rightarrow x = o \& \ldots \& x = i - x) \text{ for all } i; \]

and \( \forall x (x < i \rightarrow x = i \lor i < x) \), for all \( i. \)

14.2 Show that none of the following sentences are theorems of \( Q. \):

\[
\begin{align*}
(a) & \forall x (x \neq x'), \\
(b) & \forall x \forall y \forall z (x + z) = (y + z), \\
(c) & \forall x \forall y (x + y = y + x), \\
(d) & \forall x (x + x = x), \\
(e) & \forall x x < x', \\
(f) & \forall x \forall y (x < y \& y < x), \\
(g) & \forall x \forall y (x \cdot z) = (x \cdot y) \cdot z, \\
(h) & \forall x \forall y (x \cdot y = y \cdot x), \\
(i) & \forall x (x \cdot x = 0), \\
(j) & \forall x \forall y (x \cdot (y + z) = x \cdot y + x \cdot z).
\end{align*}
\]