Lecture 9

On Wed Oct 1, the CS faculty will be engaged in ceremonies dedicating Gates Hall and celebrating the 50th anniversary of the CS Department. Thus there will be no class this Wednesday.

Later today we will post one problem for PS3, concerned with theory Q. It will involve solving problems 14.2 of Chapter 14 in Boolos and Jeffrey. I’d like you to read the entire problem and solve parts (a), (d), (e), and (h). This will be due Wed, Oct 8. One problem will be to discuss whether Q can’t prove $\forall x. \exists y. (x < y)$ for $x < y$ defined as $\exists u. (u \neq 0 \& u + x = y)$.

The theory Q, named by at least 1950, is a finitely axiomatized weak theory of arithmetic. It has NO INDUCTION PRINCIPLE. Normally this theory is axiomatized in First-Order Logic with equality, FOL(=). We have only looked at FOL with no built in relations. So we examine FOL(=) first.

**Axioms for equality $\text{Eq}(x,y)$, in FOL**

The standard equality axioms are:

- **Reflexivity** $\forall x. \text{Eq}(x,x)$, informally $x = x$
- **Symmetry** $\forall x, y. \text{Eq}(x,y) \Rightarrow \text{Eq}(y,x)$, $x = y \Rightarrow y = x$
- **Transitivity** $\forall x, y. \text{Eq}(x,y) \Rightarrow (\text{Eq}(y,z) \Rightarrow \text{Eq}(x,y))$, i.e. $x = y \Rightarrow (y = z \Rightarrow x = z)$

We also require that all predicates respect equality, e.g. if $x = y$ and $P(x)$, then $P(y)$. This holds generally for any $R(x_1, x_2, ..., x_n)$. We state the property separately for each atomic relation and predicate of a theory.

**Realizers (Evidence extracts) for equality axioms**

- **Reflexivity** $\forall x. \text{Eq}(x,x)$ by $\lambda(x.\text{ref}())$
- **Symmetry** $\forall x, y. \text{Eq}(x,y) \Rightarrow \text{Eq}(y,x)$ by $\lambda(x.\lambda(y.\lambda(e.\text{sym}(e))))$
- **Transitivity** $\forall x, y. \text{Eq}(x,y) \Rightarrow (\text{Eq}(y,z) \Rightarrow \text{Eq}(x,y))$
  by $\lambda(x.\lambda(y.\lambda(z.\lambda(e_1.\lambda(e_2.\text{trans}(x, y, z, e_1, e_2)))))$)

The realizers give a detailed account of equality. Traditionally the constructive and intuitionistic type theories do not keep such detailed evidence. We explain why later

The theory Q also deals with these predicates and relations.
Zero(x) this predicate asserts that x is the zero element

Suc(x,y) this relation asserts that y is the successor of x

Add(x,y,z) this relation says z = x + y

Mult(x,y,z) this relation says that z = x * y

We need new axioms (not counted among the seven Q axioms) stating that these predicates and relations respect equality.

Unique Zero \( \forall x, y. \text{Zero}(x) \Rightarrow (\text{Zero}(y) \Rightarrow \text{Eq}(x, y)) \)
\[ \lambda(x.\lambda(y.\lambda(z_1.\lambda(z_2.\text{uniq}(x, y, z_1, z_2)))))) \]

Unique Successor \( \forall x, y_1, y_2. (S(x, y) \Rightarrow (S(x, y) \Rightarrow \text{Eq}(y_1, y_2))) \), we used \( S(x, y) \) for \( \text{Suc}(x, y) \) to save space.
\[ \lambda(x.\lambda(y_1.\lambda(y_2.\lambda(s_1.\lambda(s_2.\text{unisuc}(x, y_1, y_2, s_1, s_2)))))) \)

*This axiom must not be strong enough. What is a more complete axiom? The axiom would say that the successor operation respects equality*