First-Order Logic

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Chapter II

Analytic Tableaux

We now describe an extremely elegant and efficient proof procedure for propositional logic which we will subsequently extend to first order logic, and which shall be basic to our entire study. This method, which we term *analytic tableaux*, is a variant of the "semantic tableaux" of Beth [1], or of methods of Hintikka [1]. (Cf. also Anderson and Belnap [1].) Our present formulation is virtually that which we introduced in [1]. Ultimately, the whole idea derives from Gentzen [1], and we shall subsequently study the relation of analytic tableaux to the original methods of Gentzen.

§ 1. The Method of Tableaux

We begin by noting that under any interpretation the following eight facts hold (for any formulas X, Y):

1) a) If $\sim X$ is true, then X is false.
   b) If $\sim X$ is false, then X is true.
2) a) If a conjunction $X \wedge Y$ is true, then X, Y are both true.
   b) If a conjunction $X \wedge Y$ is false, then either X is false or Y is false.
3) a) If a disjunction $X \vee Y$ is true, then either X is true or Y is true.
   b) If a disjunction $X \vee Y$ is false, then both X, Y are false.
4) a) If $X \supset Y$ is true, then either X is false or Y is true.
   b) If $X \supset Y$ is false, then X is true and Y is false.

These eight facts provide the basis of the tableau method.

Signed Formulas. At this stage it will prove useful to introduce the symbols "T", "F" to our object language, and define a *signed* formula as an expression TX or FX, where X is a (unsigned) formula. (Informally, we read "TX" as "X is true" and "FX" as "X is false".)

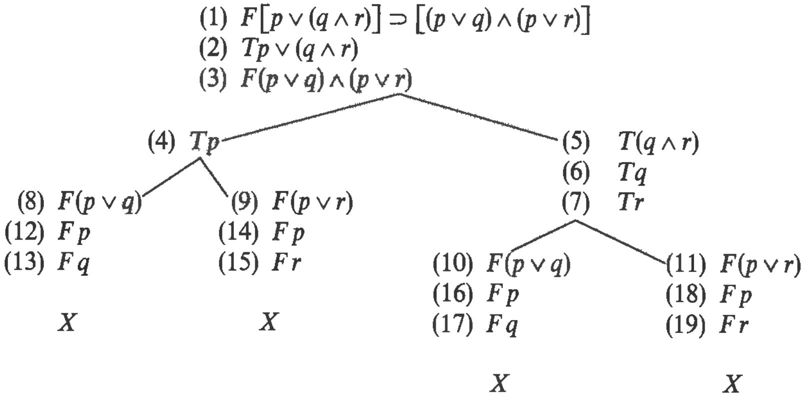
Definition. Under any interpretation, a signed formula TX is called *true* if X is true, and *false* if X is false. And a signed formula FX is called *true* if X is false, and *false* if X is true.

Thus the truth value of TX is the same as that of X; the truth value of FX is the same as that of $\sim X$.

By the *conjugate* of a signed formula we mean the result of changing "T" to "F" or "F" to "T" (thus the conjugate of TX is FX; the conjugate of FX is TX).

Illustration of the Method of Tableaux. Before we state the eight rules for the construction of tableaux, we shall illustrate the construction with an example.

Suppose we wish to prove the formula $[p \vee (q \wedge r)] \supset [(p \vee q) \wedge (p \vee r)]$. The following is a tableau which does this; the explanation is given immediately following the tableau:



Explanation. The tableau was constructed as follows. We see if we can derive a contradiction from the assumption that the formula $[p \vee (q \wedge r)] \supset [(p \vee q) \wedge (p \vee r)]$ is false. So our first line consists of this formula preceded by the letter "F". Now, a formula of the form $X \supset Y$ can be false only if X is true and Y is false. (Cf. condition B_4 of a Boolean valuation.) Thus (in the language of tableaux) TX and FY are *direct* consequences of the (signed) formula $F(X \supset Y)$. So we write the lines (2) and (3) as *direct* consequences of line (1). Now let us look at line (2); it is of the form $T(X \vee Y)$ (where $X = p$, $Y = (q \wedge r)$.) We can not draw any *direct* conclusion about the truth value of X nor about the truth value of Y; all we can infer is that *either* TX or TY. So the tableau *branches* into two columns; one for each possibility. Thus line (2) *branches* into lines (4) and (5). Line (5), viz. $T(q \wedge r)$ immediately yields Tq and Tr as direct consequences; we thus have lines (6) and (7). Now look at (3). It is of the form $F(X \wedge Y)$. This means that *either* FX *or* FY. We also know that either (4) or (5) holds. So for *each* of the possibilities (4), (5) we have one of the two possibilities FX, FY. There are hence now four possibilities. So each of the branches (4), (5) branches again into the possibilities FX, FY. More specifically, (4) branches to (8), (9), and (5) branches to (10), (11) (which are respectively the same as (8), (9)). Lines (12), (13) are direct consequences of (8); (14), (15) are direct consequences of (9); (16), (17) of (10); and (18), (19) of (11).

We now look at the leftmost branch and we shall see that (12) is a direct contradiction of (4) (i.e. it is the conjugate of (4)), so we put a cross after (13) to signify that this branch leads to a contradiction. Similarly, (14) contradicts (4), so we "close" the branch leading to (15)—i.e. we put a cross after (15). The next branch is closed by virtue of (17) and (6). Finally, the rightmost branch is closed by virtue of (19) and (7). Thus all branches lead to a contradiction, so line (1) is untenable. Thus $[p \vee (q \wedge r)] \supset [(p \vee q) \wedge (p \vee r)]$ can never be false in any interpretation, so it is a tautology.

Remarks. (i) The numbers put to the left of the lines were only for the purpose of identification in the above explanations; we do not need them for the actual construction.

(ii) We could have closed some of our branches a bit earlier; lines (13), (15) are superfluous. In subsequent examples we shall close a branch as soon as a contradiction appears (a contradiction that is of the form of two formulas FX, TX).

Rules for the Construction of Tableaux. We now state all the rules in schematic form; explanations immediately follow. For each logical connective there are two rules; one for a formula preceded by "T", the other for a formula preceded by "F":

$$1)\quad \frac{T{\sim}X}{FX} \qquad \frac{F{\sim}X}{TX}$$

$$2)\quad \frac{T(X \wedge Y)}{\begin{array}{c} TX \\ TY \end{array}} \qquad \frac{F(X \wedge Y)}{FX \,|\, FY}$$

$$3)\quad \frac{T(X \vee Y)}{TX \,|\, TY} \qquad \frac{F(X \vee Y)}{\begin{array}{c} FX \\ FY \end{array}}$$

$$4)\quad \frac{T(X \supset Y)}{FX \,|\, TY} \qquad \frac{F(X \supset Y)}{\begin{array}{c} TX \\ FY \end{array}}$$

Some Explanations. Rule 1) means that from $T{\sim}X$ we can directly infer FX (in the sense that we can subjoin FX to any branch passing through $T{\sim}X$) and that from $F{\sim}X$ we can directly infer TX. Rule 2) means that $T(X \wedge Y)$ directly yields both TX, TY, whereas $F(X \wedge Y)$ *branches* into FX, FY. Rules 3) and 4) can now be understood analogously.

Signed formulas, other than signed variables, are of two types; (A) those which have *direct* consequences (viz. $F{\sim}X$, $T{\sim}X$, $T(X \wedge Y)$, $F(X \vee Y)$, $F(X \supset Y)$); (B) those which *branch* (viz. $F(X \wedge Y)$, $T(X \vee Y)$, $T(X \supset Y)$).

It is practically desirable in constructing a tableau, that when a line of type (A) appears on the tableau, we simultaneously subjoin its consequences to *all* branches which pass through that line. Then that line need never be used again. And in using a line of type (B), we divide *all* branches which pass through that line into sub-branches, and the line need never be used again. For example, in the above tableau, we use (1) to get (2) and (3), and (1) is never used again. From (2) we get (4) and (5), and (2) is never used again. Line (3) yields (8), (9), (10), (11) and (3) is never used again, etc.

If we construct a tableau in the above manner, it is not difficult to see, that after a finite number of steps we must reach a point where every line has been used (except of course, for signed variables, which are never used at all to create new lines). At this point our tableau is *complete* (in a precise sense which we will subsequently define).

One way to complete a tableau is to work systematically downwards i.e. never to use a line until all lines above it (on the same branch) have been used. Instead of this procedure, however, it turns out to be more efficient to give priority to lines of type (A)—i.e. to use up all such lines at hand before using those of type (B). In this way, one will omit repeating the same formula on different branches; rather it will have only one occurrence *above* all those branch points.

As an example of both procedures, let us prove the formula $[p \supset (q \supset r)] \supset [(p \supset q) \supset (p \supset r)]$. The first tableau works systematically downward; the second uses the second suggestion. For the convenience of the reader, we put to the right of each line the number of the line from which it was inferred.

First Tableau

<table>
<tr><td colspan="6">(1) $F[p \supset (q \supset r)] \supset [(p \supset q) \supset (p \supset r)]$</td></tr>
<tr><td colspan="6">(2) $Tp \supset (q \supset r)$ (1)</td></tr>
<tr><td colspan="6">(3) $F(p \supset q) \supset (p \supset r)$ (1)</td></tr>
<tr><td colspan="2">(4) Fp (2)</td><td colspan="4">(5) $T(q \supset r)$ (2)</td></tr>
<tr><td colspan="2">(6) $T(p \supset q)$ (3)</td><td colspan="4">(8) $T(p \supset q)$ (3)</td></tr>
<tr><td colspan="2">(7) $F(p \supset r)$ (3)</td><td colspan="4">(9) $F(p \supset r)$ (3)</td></tr>
<tr><td>(10) Fp (6)</td><td>(11) Tq (6)</td><td colspan="2">(14) Fq (5)</td><td colspan="2">(15) Tr (5)</td></tr>
<tr><td>(12) Tp (7)</td><td>(13) Tp (7)</td><td>(16) Fp (8)</td><td>(17) Tq (8)</td><td>(18) Fp (8)</td><td>(19) Tq (8)</td></tr>
<tr><td>X</td><td>X</td><td>(20) Tp (9)</td><td>X</td><td>(21) Tp (9)</td><td>(22) Tp (9)</td></tr>
<tr><td></td><td></td><td>X</td><td></td><td>X</td><td>(23) Fr (9)</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td>X</td></tr>
</table>

Second Tableau

(1) $F[p \supset (q \supset r)] \supset [(p \supset q) \supset (p \supset r)]$

(2) $Tp \supset (q \supset r)$ (1)

(3) $F(p \supset q) \supset (p \supset r)$ (1)

(4) $T(p \supset q)$ (3)

(5) $F(p \supset r)$ (3)

(6) Tp (5)

(7) Fr (5)

(8) Fp (2)	(9) $T(q \supset r)$ (2)		
X	(10) Fp (4)	(11) Tq (4)	
	X	(12) Fq (9)	(13) Tr (9)
		X	X

It is apparent that Tableau (2) is quicker to construct than Tableau (1), involving only 13 rather than 23 lines.

As another practical suggestion, one might put a check mark to the right of a line as soon as it has been used. This will subsequently aid the eye in hunting upward for lines which have not yet been used. (The check marks may be later erased, if the reader so desires.)

The method of analytic tableaux can also be used to show that a given formula is a truth functional consequence of a given finite set of formulas. Suppose we wish to show that $X \supset Z$ is a truth-functional consequence of the two formulas $X \supset Y$, $Y \supset Z$. We could, of course, simply show that $[(X \supset Y) \wedge (Y \supset Z)] \supset (X \supset Z)$ is a tautology. Alternatively, we can construct a tableau starting with

$$T(X \supset Y),$$
$$T(Y \supset Z),$$
$$F(X \supset Z)$$

and show that all branches close.

In general, to show that Y is truth-functionally implied by $X_1, \ldots, X_n$, we can construct either a closed analytic tableau starting with $F(X_1 \wedge \ldots \wedge X_n) \supset Y$, or one starting with

$$\begin{array}{c} TX_1 \\ \vdots \\ TX_n \\ FY \end{array}$$

Tableaux using unsigned formulas. Our use of the letters "T" and "F", though perhaps heuristically useful, is theoretically quite dispensable—simply delete every "T" and substitute "$\sim$" for "F". (In which case, incidentally, the first half of Rule 1) becomes superfluous.) The rules then become:

$$1)\quad \frac{\sim\sim X}{X}$$

$$2)\quad \frac{X \wedge Y}{\begin{array}{c} X \\ Y \end{array}} \qquad \frac{\sim(X \wedge Y)}{\sim X \mid \sim Y}$$

$$3)\quad \frac{X \vee Y}{X \mid Y} \qquad \frac{\sim(X \vee Y)}{\begin{array}{c} \sim X \\ \sim Y \end{array}}$$

$$4)\quad \frac{X \supset Y}{\sim X \mid Y} \qquad \frac{\sim(X \supset Y)}{\begin{array}{c} X \\ \sim Y \end{array}}$$

In working with tableaux which use unsigned formulas, "closing" a branch naturally means terminating the branch with a cross, as soon as two formulas appear, one of which is the *negation* of the other. A tableau is called *closed* if every branch is closed.

By a tableau *for* a formula X, we mean a tableau which starts with X. If we wish to prove a formula X to be a tautology, we construct a tableau not for the formula X, but for its negation $\sim X$.

A Unifying Notation. It will save us considerable repetition of essentially the same arguments in our subsequent development if we use the following unified notation which we introduced in [2].

We use the letter "α" to stand for any signed formula of type A—i.e. of one of the five forms $T(X \wedge Y)$, $F(X \vee Y)$, $F(X \supset Y)$, $T\sim X$, $F\sim X$. For every such formula α, we *define* the two formulas α_1 and α_2 as follows:

If $\alpha = T(X \wedge Y)$, then $\alpha_1 = TX$ and $\alpha_2 = TY$.
If $\alpha = F(X \vee Y)$, then $\alpha_1 = FX$ and $\alpha_2 = FY$.
If $\alpha = F(X \supset Y)$, then $\alpha_1 = TX$ and $\alpha_2 = FY$.
If $\alpha = T\sim X$, then $\alpha_1 = FX$ and $\alpha_2 = FX$.
If $\alpha = F\sim X$, then $\alpha_1 = TX$ and $\alpha_2 = TX$.

For perspicuity, we summarize these definitions in the following table:

α	α_1	α_2
$T(X \wedge Y)$	TX	TY
$F(X \vee Y)$	FX	FY
$F(X \supset Y)$	TX	FY
$T{\sim}X$	FX	FX
$F{\sim}X$	TX	TX

We note that under any interpretation, α is true iff α_1, α_2 are *both* true. Accordingly, we shall also refer to an α as a formula of *conjunctive* type.

We use "β" to stand for any signed formula of type B—i.e. one of the three forms $F(X \wedge Y)$, $T(X \vee Y)$, $T(X \supset Y)$. For every such formula β, we define the two formulas β_1, β_2 as in the following table:

β	β_1	β_2
$F(X \wedge Y)$	FX	FY
$T(X \vee Y)$	TX	TY
$T(X \supset Y)$	FX	TY

In any interpretation, β is true iff *at least one* of the pair β_1, β_2 is true. Accordingly, we shall refer to any β-type formula as a formula of *disjunctive* type.

We shall sometimes refer to α_1 as the *first component* of α and α_2 as the *second component* of α. Similarly, for β.

By the *degree* of a signed formula TX or FX we mean the degree of X. We note that α_1, α_2 are each of lower degree than α, and β_1, β_2 are each of lower degree than β. Signed variables, of course, are of degree 0.

We might also employ an α, β classification of *unsigned* formulas in an analogous manner, simply delete all "T", and replace "F" by "$\sim$". The tables would be as follows:

α	α_1	α_2
$X \wedge Y$	X	Y
${\sim}(X \vee Y)$	${\sim}X$	${\sim}Y$
${\sim}(X \supset Y)$	X	${\sim}Y$
${\sim}{\sim}X$	X	X

β	β_1	β_2
${\sim}(X \wedge Y)$	${\sim}X$	${\sim}Y$
$X \vee Y$	X	Y
$X \supset Y$	${\sim}X$	Y

Summary

Rule A
$$\begin{array}{c}\alpha\\ \hline \alpha_1\\ \alpha_2\end{array}$$

Rule B
$$\begin{array}{c|c}\multicolumn{2}{c}{\beta}\\ \hline \beta_1 & \beta_2\end{array}$$

exactly one of $X, \bar{X}$ is in S) is a truth set. Show that any *upward* closed set satisfying (0) is a truth set.

Precise Definition of Tableaux. We have deliberately waited until the introduction of our unified notation in order to give a precise definition of an analytic tableau, since the definition can now be given more compactly.

Definition. An analytic tableau for X is an ordered dyadic tree, whose points are (occurrences of) formulas, which is constructed as follows. We start by placing X at the origin. Now suppose $\mathscr{T}$ is a tableau for X which has already been constructed; let Y be an end point. Then we may extend $\mathscr{T}$ by either of the following two operations.

(A) If some α occurs on the path P_Y, then we may adjoin either α_1 or α_2 as the sole successor of Y. (In practice, we usually successively adjoin α_1 and then α_2.)

(B) If some β occurs on the path P_Y, then we may simultaneously adjoin β_1 as the left successor of Y and β_2 as the right successor of Y.

The above inductive definition of tableau for X can be made explicit as follows. Given two ordered dyadic trees $\mathscr{T}_1$ and $\mathscr{T}_2$, whose points are occurrences of formulas, we call $\mathscr{T}_2$ a *direct extension* of $\mathscr{T}_1$ if $\mathscr{T}_2$ can be obtained from $\mathscr{T}_1$ by one application of the operation (A) or (B) above. Then $\mathscr{T}$ is a tableau for X iff there exists a finite sequence $(\mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_n = \mathscr{T})$ such that $\mathscr{T}_1$ is a 1-point tree whose origin is X and such that for each $i < n, \mathscr{T}_{i+1}$ is a direct extension of $\mathscr{T}_i$.

To repeat some earlier definitions (more or less informally stated) a branch θ of a tableau for signed (unsigned) formulas is *closed* if it contains some signed formula and its conjugate (or some unsigned formula and its negation, if we are working with unsigned formulas.) And $\mathscr{T}$ is called closed if every branch of $\mathscr{T}$ is closed. By a *proof* of (an unsigned formula) X is meant a closed tableau for $F\,X$ (or for $\sim X$, if we work with unsigned formulas.)

Exercise. By the tableau method, prove the following tautologies:

(1) $q \supset (p \supset q)$

(2) $((p \supset q) \wedge (q \supset r)) \supset (p \supset r)$

(3) $((p \supset q) \wedge (p \supset r)) \supset (p \supset (q \wedge r))$

(4) $[((p \supset r) \wedge (q \supset r)) \wedge (p \vee q)] \supset r$

(5) $\sim (p \wedge q) \supset (\sim p \vee \sim q)$

(6) $\sim (p \vee q) \supset (\sim p \wedge \sim q)$

(7) $(\sim p \vee \sim q) \supset \sim (p \wedge q)$

(8) $(p \vee (q \wedge r)) \supset ((p \vee q) \wedge (p \vee r))$

§ 2. Consistency and Completeness of the System

Consistency. It is intuitively rather obvious that any formula provable by the tableau method must be a tautology—equivalently, given any closed tableau, the origin must be unsatisfiable. This intuitive conviction can be justified by the following argument.

Consider a tableau $\mathscr{T}$ and an interpretation v_0 whose domain includes at least all the variables which occur in any point of $\mathscr{T}$. Let us call a branch θ of $\mathscr{T}$ *true* under v_0 if *every* term of θ is true under v_0. And we shall say that the tableau $\mathscr{T}$ (as a whole) is true under v_0 iff at least one branch of $\mathscr{T}$ is true under v_0.

The next step is to note that if a tableau $\mathscr{T}_2$ is an immediate extension of $\mathscr{T}_1$, then $\mathscr{T}_2$ must be true in every interpretation in which $\mathscr{T}_1$ is true. For if $\mathscr{T}_1$ is true, it must contain at least one true branch θ. Now $\mathscr{T}_2$ was obtained from $\mathscr{T}_1$ by adding one or two successors to the end point of some branch θ_1 of $\mathscr{T}_1$; if θ_1 is distinct from θ, then θ is still a branch of $\mathscr{T}_2$, hence $\mathscr{T}_2$ contains the true branch θ, so $\mathscr{T}_2$ is true. On the other hand, suppose θ is identical with θ_1—i.e. suppose θ is the branch of $\mathscr{T}_1$ which was extended in $\mathscr{T}_2$. If θ was extended by operation (*A*), then some α appears as a term in θ, and θ has been extended either to (θ_1, α_1) or to (θ_1, α_2), so either (θ_1, α_1) or (θ_1, α_2) is a branch of $\mathscr{T}_2$. But α_1, α_2 are both true since α is, hence $\mathscr{T}_2$ contains the true branch (θ_1, α_1) or (θ_1, α_2). If θ was extended by operation (*B*), then some β occurs in θ and both (θ_1, β_1) and (θ_1, β_2) are branches of $\mathscr{T}_2$. But since β is true, then at least one of β_1, β_2 is true, hence one of the branches (θ_1, β_1) or (θ_1, β_2) of $\mathscr{T}_2$ is true, so again $\mathscr{T}_2$ is true.

We have thus shown that any immediate extension of a tableau which is true (under a given interpretation) is again true (under the given interpretation). From this it follows by mathematical induction that for any tableau $\mathscr{T}$, if the origin is true under a given interpretation v_0, then $\mathscr{T}$ must be true under v_0. Now a closed tableau $\mathscr{T}$ obviously cannot be true under any interpretation, hence the origin of a closed tableau cannot be true under any interpretation—i.e. the origin of any closed tableau must be unsatisfiable. From this it follows that every formula provable by the tableau method must be a tautology. It therefore further follows that the tableau method is *consistent* in the sense that no formula and its negation are both provable (since no formula and its negation can both be tautologies).

Completeness. We now consider the more delicate converse situation: Is every tautology provable by the method of tableaux? Stated otherwise, if X is a tautology, can we be sure that there exists at least one *closed* tableau starting with FX? We might indeed ask the following bolder question: If X is a tautology, then will *every* complete tableau for FX

close? An affirmative answer to the second question would, of course, be even better than an affirmative answer to the first, since it would mean that any single completed tableau $\mathcal{T}$ for FX would decide whether X is a tautology or not.

Before the reader answers the question too hastily, we should consider the following. If we delete some of the rules for the construction of tableaux, it will *still* be true that a *closed* tableau for FX *always* indicates that X is a tautology. But if we delete too many of the rules, then we may not have left enough power to always derive a closed tableau for FX whenever X is a tautology. (For example, if we delete the first half of the conjunction rule, then it would be impossible to prove the tautology $(p \wedge q) \supset p$, though it would still be possible to prove $p \supset [q \supset (p \wedge q)]$. If we delete the second half but retain the first half, then we could prove the first tautology above, but not the second.) The question, therefore, is whether our present set of rules is *sufficient* to do this. Our present purpose is to show that they are sufficient.

We shall give the proof for tableaux using signed formulas (the modifications for tableaux using unsigned formulas are obvious—or indeed the result for tableaux for unsigned formulas follows directly from the result for tableaux with signed formulas.)

We are calling a branch θ of a tableau *complete* if for every α which occurs in θ, both α_1 and α_2 occur in θ, and for every β which occurs in θ, at least one of β_1, β_2 occurs in θ. We call a tableau $\mathcal{T}$ *completed* if every branch of $\mathcal{T}$ is either closed or complete. We wish to show that if $\mathcal{T}$ is any *completed* open tableau (open in the sense that at least one branch is not closed), then the origin of $\mathcal{T}$ is satisfiable. More generally, we shall show

Theorem 1. *Any complete open branch of any tableau is (simultaneously) satisfiable.*

We shall actually prove something stronger. Suppose θ is a *complete open* branch of a tableau $\mathcal{T}$; let S be the *set* of terms of θ. Then the set S satisfies the following three conditions (for every α, β):

H_0: No signed *variable* and its conjugate are both in S[1].

H_1: If $\alpha \in S$, then $\alpha_1 \in S$ and $\alpha_2 \in S$.

H_2: If $\beta \in S$, then $\beta_1 \in S$ or $\beta_2 \in S$.

Sets S—whether finite or infinite—obeying conditions H_0, H_1, H_2 are of fundamental importance—we shall call them *Hintikka sets* (after Hintikka who studied their properties explicitly). We shall also refer to

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[1] Indeed no signed formula and its conjugate appear in S, but we do not need to involve this stronger fact.

Hintikka sets as sets which are *saturated downwards.* We shall also call any finite or denumerable *sequence* θ a *Hintikka sequence* if its set of terms is a Hintikka set.

Let us pause for a moment to compare the notion of *downward* saturation with that of saturation discussed earlier (cf. the preceding section on truth sets re-visited). The definition of a saturated set differs from that of a Hintikka set in that in H_1, H_2 "if" is replaced by "if and only if", and H_0 is strengthened to condition (0). So every saturated set is obviously also a Hintikka set. But a Hintikka set need not be saturated (e.g. any set of signed *variables* which contains no signed variable and its conjugate vacuously satisfies H_1, H_2, but such a set is certainly not saturated.)

Theorem 1 is substantially to the effect that every *finite* Hintikka set S is satisfiable. The finiteness of S, however, is not needed in the proof (nor does it even simplify the proof), so we shall prove

Hintikka's Lemma. *Every downward saturated set S (whether finite or infinite) is satisfiable.*

We remark that Hintikka's lemma is equivalent to the statement that every Hintikka set can be extended to a (i.e. is a subset of some) *saturated* set. We remark that Hintikka's lemma also holds for sets of *unsigned* formulas (where by a Hintikka set of unsigned formulas we mean a set S satisfying H_1, H_2 and in place of H_0, the condition that no variable and its negation are both elements of S).

Proof of Hintikka's Lemma. Let S be a Hintikka set. We wish to find an interpretation in which every element of S is true. Well, we assign to each variable p, which occurs in at least one element of S, a truth value as follows:

(1) If $Tp \in S$, give p the value true.

(2) If $Fp \in S$, give p the value false.

(3) If neither Tp nor Fp is an element of S, then give p the value true or false at will (for definiteness, let us suppose we give it the value true.)

We note that the directions (1), (2) are compatible, since no Tp and Fp both occur in S (by hypothesis H_0). We now show that every element of S is true under this interpretation. We do this by induction on the degree of the elements.

It is immediate that every signed *variable* which is an element of S is true under this interpretation (the interpretation was constructed to insure just this). Now consider an element X of S of degree greater than 0, and suppose all elements of S of lower degree than X are true. We wish to show that X must be true. Well, since X is of degree greater than zero, it must be either some α or some β.

Case 1. Suppose it is an α. Then α_1, α_2 must also be in S (by H_1). But α_1, α_2 are of lower degree than α. Hence by inductive hypothesis α_1 and α_2 are both true. This implies that α must be true.

Case 2. Suppose X is some β. Then at least one of β_1, β_2 is in S (by H_2). Whichever one is in S, being of lower degree than β, must be true (by inductive hypothesis). Hence β must be true. This concludes the proof.

Remark. If we hadn't used the unifying "α, β" notation, we would have had to analyze eight cases rather than two.

Having proved Hintikka's lemma, we have, of course, also proved Theorem 1. This in turn implies

Theorem 2. *(Completeness Theorem for Tableaux)*

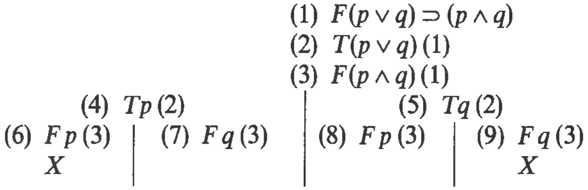
(*a*) *If X is a tautology, then every completed tableau starting with FX must close.*

(*b*) *Every tautology is provable by the tableau method.*

To derive statement (*a*) from Theorem 1, suppose $\mathcal{T}$ is a complete tableau starting with FX. If $\mathcal{T}$ is open, then FX is satisfiable (by Theorem 1), hence X cannot be a tautology. Hence if X is a tautology then $\mathcal{T}$ must be closed.

Let us note that for S a finite Hintikka set, the proof of Hintikka's lemma effectively gives us an interpretation which satisfies S. Therefore, if X is not a tautology, then a completed tableau for FX provides us with a counterexample to X (i.e. an interpretation in which X is false).

Example. Let X be the formula $(p \vee q) \supset (p \wedge q)$. Let us construct a completed tableau for FX:



This tableau has two open branches. Let us consider the branch whose end point is (7). Acording to the method of Hintikka's proof, if we declare p true and q false, we have an interpretation which satisfies all lines of this branch. The reader can verify this by successively showing that (3), (2), (1) are true under this interpretation. Hence $F(p \vee q) \supset (p \wedge q)$ is true under this interpretation, which means $(p \vee q) \supset (p \wedge q)$ is false under this interpretation. Likewise the open branch terminating in (8) gives us another interpretation (viz. q is true, p is false) which is a counterexample to X.

Tableaux for Finite Sets. If S is a *finite* set $\{X_1, \ldots, X_n\}$, by a tableau for S is meant a tableau starting with

$$\begin{array}{c} X_1 \\ X_2 \\ \vdots \\ X_n \end{array}$$

and then continued using Rules A, B.

We leave it to the reader to modify our previous arguments for tableaux for single formulas, and prove:

Theorem. *A finite set S is unsatisfiable iff there exists a closed tableau for S.*

We shall consider tableaux for *infinite* sets in a subsequent chapter.

Exercise. There is another way of proving the completeness theorem which does not use Hintikka's lemma.

Show (without use of the completeness theorem): (1) If there exists a closed tableau for $S \cup \{\alpha_1, \alpha_2\}$ then there exists a closed tableau for $S \cup \{\alpha\}$; (2) if there exist closed tableaux for $S \cup \{\beta_1\}$ and for $S \cup \{\beta_2\}$, then there exists a closed tableau for $S \cup \{\beta\}$; (3) if all elements of S are of degree 0, and S is unsatisfiable, then there exists a closed tableau for S (trivial!).

Now define the degree of a finite set S to be the sum of the degrees of the elements of S. Using (1), (2), (3) above, show by induction on the degree of S that if S is unsatisfiable, then there exists a closed tableau for S.

Atomically Closed Tableaux. Let us call a tableau *atomically* closed if every branch contains some *atomic* element and its conjugate. [By an *atomic* element we mean a propositional variable, if we are working with unsigned formulas, and a *signed* propositional variable if we are working with signed formulas. If we are working with unsigned formulas, then by an atomically closed tableau we mean a tableau in which every branch contains some propositional variable and its negation.]

Suppose we construct a completed tableau $\mathcal{T}$ for a set S, and declare a branch "closed" only if it is atomically closed. Now suppose $\mathcal{T}$ contains an (atomically) open branch B. Then the set of elements of B is still a Hintikka set (because condition H_0 requires only that the set contain no *atomic* elements and its conjugate), hence is satisfiable (by Hintikka's lemma). We thus have:

Theorem. *If S is unsatisfiable, then there exists an atomically closed tableau for S.*

Corollary. *If there exists a closed tableau for S, then there exists an atomically closed tableau for S.*