Problem Set 5

Handed out: 4/28/21; due: 5/12/21

1. Consider the policy of vaccinating healthcare workers, long-term care residents, frontline essential workers, and people over 75 first.

   (a) Carefully define a state space, outcome space, probability, and utility such that this policy maximizes expected utility. The answer here isn't unique (in fact, there are many possible answers), but we do expect you to make reasonable choices and reasonable simplifying assumptions. You must justify your choices (e.g., whatever your state space is, explain why it's a reasonable choice). Also, make clear the simplifying assumptions you're making and how they affect the choice.

   (b) Can we justify this policy in terms of QALYs?

2. [GRAD:] Consider the Anscombe-Aumann (AA) framework, defined on pp. 38-89 of Kreps, where an AA act is a function from a state space S to lotteries (i.e., distributions over prizes). This gives a mixture space: if a and b are two AA acts, then $\alpha a + (1-\alpha)b$ is the act that in state s, gives the lottery $\alpha a(s) + (1-\alpha)b(s)$. Recall that a *capacity* W on S maps subsets of S to $[0,1]$, and $W(\emptyset) = 0$, $W(S) = 1$, and $W(A) \leq W(B)$ if $A \subseteq B$. (This was defined in Prof. Halpern's notes in the context of cumulative prospect theory.) Suppose that f is a function whose values on $f : S \rightarrow \{x_1, \ldots, x_n\}$, where $x_i > x_j$ if $i > j$. Define a notion of expectation for capacities by taking $E_W(f) = x_1 + \sum_{i=2}^{n} W(f \geq x_i)(x_i - x_{i-1})$.

   (a) Show that if Pr is a probability measure, then

   $$E_{\Pr}(f) = f(x_1) + \sum_{i=2}^{n} \Pr(f \geq x_i)(x_i - x_{i-1}).$$

   (Thus, the definition of expectation for capacities generalizes that for probabilities.)

   (b) Given a capacity W, define an order $\succ_W$ on acts by taking $a \succ_W b$ iff $E_W(a) > E_W(b)$. Show that $\succ_W$ is a preference order.

(c) Show that $\succ_W$ does not in general satisfy independence.

(d) Show that $\succ_W$ does satisfy comonotonic independence (as defined on slide 11 of Prof. Halpern's slides on problems with maximizing expected utility).

3. In this exercise, we show that if we allow randomization in programs (so that if a and b are programs, then so is $\alpha a + (1-\alpha)b$, where $\alpha \in [0,1]$), then a variant of the Cancellation postulate gives us independence for rational coefficients. Note that just as a program without randomization can viewed as a function from truth assignments to primitive programs, a program with randomization can be viewed as a function from from truth assignments to distributions overprimitive programs. For example, if t is the only test, and p_1, p_2, and p_3 are primitive programs, then $a = \frac{1}{3}p_1 + \frac{2}{3}(\textbf{if } t \textbf{ then } p_2 \textbf{ else } p_3)$ can be identified with the function f_a such that

   - $f_a(t)(p_1) = 1/3$; $f_a(t)(p_2) = 2/3$
   - $f_a(\neg t)(p_1) = 1/3$; $f_a(\neg t)(p_3) = 2/3$.

   Now consider the following variant of Cancellation:

   If $\langle a_1, \ldots, a_n \rangle$ and $\langle b_1, \ldots, b_n \rangle$ are two sequences of programs (with randomization) such that, for all primitive programs p_i and all truth assignments v, we have $f_{a_1}(v)(p_i) + \cdots + f_{a_n}(v)(p_i) = f_{b_1}(v)(p_i) + \cdots + f_{b_n}(v)(p_i)$, then if, for some $k < n$, we have that $a_i \succeq b_i$ for $i \leq k$ and $a_{k+1} = \cdots = a_n$ and $b_{k+1} = \cdots = b_n$, then $b_n \succeq a_n$.

   Show that it follows that $a \succeq b$ iff for all c and all rational $\alpha \in (0,1]$, we have $\alpha a + (1-\alpha)c \succeq \alpha b + (1-\alpha)c$.

4. The *Borda count* constructs a social ranking out of individual rankings as follows: Suppose that there are k alternatives. Each individual submits a ranking of the alternatives from first to kth. (For the purposes of this problem, you can assume that the ranking is strict; there are no ties.) An alternative gets k points for every first place vote, $k-1$ for every second place vote, and so on. Alternatives are then ranked by point totals.

   (a) Which of the assumptions in Arrow's theorem are violated?

   (b) Give an example to demonstrate your claim.

(Note: we have not done Arrow's theorem yet. We will cover it next week.)

5. Imagine that a finite set of alternatives are ordered $a_1, \dots, a_K$. A preference relation is said to be *single-peaked* with respect to this order if there is an k^* such that if $l < k \leq k^*$ it is not the case that $a_l \succ a_k$, and if $n \leq k < l$ then it is not the case that $a_l \succ a_k$. (If you were to draw this as a graph, you think that things get better until they hit a peak a a_{k^*}, then they get worse. Intuitively, the further a way an option is from the peak, the worse it is.)

   (a) Draw a graph of a utility function representing sinle-peaked preferences for the following three cases: $k^* = 1$, $k^* = K$, and $1 < k^* < K$. Draw a graph of a utility function that does not represent single-peaked preferences.

   (b) Suppose that all voters have single-peaked preferences and let $p(k)$ denote the fraction of voters whose peaks are a_k. p_k describes a probability distribution. Show that if a_m is a median of this distribution, then there is no alternative b that gets more votes than a_m in an election between a_m and b. (You may assume either that a voter who is tied between two alternatives votes for both or that a tied voter votes for neither.)

   (c) Define $a \succ b$ iff at least as many voters prefer a over b in an election between the two. Call this the majority-rule procedure. Show that $\succ$ is a preference relation if all voters have single-peaked preferences, but show by example that this may not be the case if preferences are not single-peaked.

   (d) Part (c) shows that the majority-rule procedure does not satisfy the universal domain axiom: it does not map an arbitrary tuple of rankings to a social preference relation. Show that the majority-rule procedure (whether single-peaked or not) satisfies all of Arrow's axioms other than the universal domain axiom.

(Note: again, we won't cover this material until next week.)