Computer Science 5846: HW2

Handed out: March 3, 2021. Due: March 17 at 9:30 AM, 2021. (If you hand it in late, but before Friday, Mar. 19, at 3 PM, you will get a 15% penalty.) If you are enrolled in CS5846 or ECON6760, you have to do all the questions. If you are in ECON3810, you do not have to do the questions labeled "GRAD".

1. Prove that if f is a strictly increasing function, then $\text{maximin}(u) = \text{maximin}(f(u))$. Show by example that $opt^\alpha(u)$ may not be the same as $opt^\alpha(f(u))$, and that $regret(u)$ may not be the same as $regret(f(u))$.

2. Prove that if f is a positive affine transformation, then $opt^\alpha(u) = opt^\alpha(f(u))$ and $regret(u) = regret(f(u))$. (It follows from the previous question that we also have $maximin(u) = maximin(f(u)$, although you don't have to prove this again.)

3. (a) Suppose that you are an insurance broker, trying to decide whether to offer fire insurance to a business located in an old wooden structure with an oil furnace in the basement. What is a reasonable state space and outcome space for the decision? How do things change if you have reason to believe that the owner of the business was a convicted arsonist? (There's no "right" or "wrong" answer here. But you should argue for the reasonableness of your choices.)

   (b) Recall the "better red than dead" example from Prof. Halpern's notes. Define a reasonable state space that is independent of the acts "arm and "disarm". (The outcome space is still "dead", "red", "status quo", and "improved society".) Again, there is no unique "right" answer. But whatever answer you give, you should argue that the acts are independent of the states you choose.

4. Recall the algorithm for the location problem: The robot goes to $+1$, then -2, then $+4$, then -8, $\ldots$, until it finds the object.

   (a) Show that this algorithm has a competitive ratio of 9. That is, show that, no matter where the object is located, it will take the robot at most 9 times as many steps to find the object using this algorithm as it would to go directly to the object. [Hint: first consider the points where the robot changes direction—1,-2,4,-8, ...—and figure out how many steps it takes the robot to get to the nth change point. Then consider arbitrary points.]

(b) Show that if $k < 9$, then for all c, there is a location N_c for the object such that the robot will take more than $kN_c + c$ steps to find the object using this algorithm.

5. Suppose that instead of increasing by powers of 2, we increase by powers of 3, so that the robot goes from +1 to -3 to 9, and so on.

   (a) Show that the competitive ratio is now 10, not 9.

   (b) GRAD: compute the competitive ratio if we increase by powers of x, for $x > 1$. Use calculus to show that the optimal choice is to take $x = 2$.

6. Consider (a slight variant of) the ski rental problem from the notes. Suppose that you will be skiing for an unknown number of days, but at most N and at least one. You initially don't have skis. Each time you go skiing, if you haven't already bought skis, you can choose to either rent or buy skis. Renting skis costs $\$r$/day; buying them costs $\$p$. Assume that whether or not you buy skis does not affect how many days you ski; that is, you will be skiing for d days out of N, independent of whether you buy or rent skis. (Note: this is almost certainly false for many people. Once they buy skis, they view it as a commitment and are less likely to quit skiing. The assumption makes sense only if you view someone else as deciding whether or not you ski; your only decision is whether to buy or rent.) For simplicity, suppose that your utility is determined completely by how much money you spend. (You are indifferent about skiing and the choice of whether or not you go is not up to you, but determined externally; your only choice is whether to rent or buy.) With these (admittedly unreasonable) assumptions, model the ski rental problem as a decision problem.

   (a) Describe the states, acts, and outcomes for this problem.

   (b) Which act(s) are optimal according to the maximin rule?

   (c) Which act(s) are optimal acording to the minimax regret rule?

   (d) GRAD: Which act(s) give the optimal competitive ratio?

   (e) Explain clearly why the assumption that your decision to buy or rent does not affect whether you ski is necessary in your analysis.

(f) **Challenge problem:** (You don't have to hand this in, and there's no extra credit for it. This is just something you might find fun to think about.) Suppose that $p - r < Nr - p$. Show that randomization helps in terms of minimax regret. That is, show that there is a randomizaed act whose minimax regret is better than any deterministic act.

Note: the act(s) that are optimal depend on the values of the parameters p, r, and N. (Actually, all that matters is p/r and N.)

7. We showed in the notes that regret suffers from the problem of irrelevant alternatives. Specificially, we gave an example with two acts a_1 and a_2 and two states s_1 and s_2 such that a_1 has lower regret than a_2, but if a third act is added, then a_2 has lower regret than a_1. Show that, in this example, this cannot happen with expected regret, no matter what the probability distribution on s_1 and s_2. Thus, minimizing expected regret does not suffer from the addition of "irrelevant alternatives".

8. Show that $a \succ^3_{\mathcal{P}} a'$ implies $a \succ^4_{\mathcal{P}} a'$, where $\mathcal{P}$ is a set of probability distributions and $\succ^3_{\mathcal{P}}$ and $\succ^4_{\mathcal{P}}$ are the partial orders defined in Prof. Halpern's notes.

9. GRADUATE: Find a variant of Sen's β that, together with α, characterizes those choice functions C such that C is determined by a strict partial order $\succ$ (i.e., $\succ$ is irreflexive and transitive). That is, find a variant β' of β such that, given a partial order $\succ$, the function $C_\succ$ defined as in the class notes is a choice function that satisfies α and β' and, conversely, if a choice function C satisfies α and β', then $C = C_\succ$ for some partial order $\succ$. (Note that this variant may not be unique; we'll take anything reasonable that works.) [Hint: consider 2-element subsets.]