Expected Utility

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motivating examples
powerball
# Powerball Expected Payout

<table>
<thead>
<tr>
<th>Numbers Matched</th>
<th>Prize</th>
<th>Prize - Cost</th>
<th>Likelihood</th>
<th>Probability</th>
<th>(Prize - Cost) x Probability</th>
<th>Expected Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 white and red</td>
<td>$450,000,000</td>
<td>$449,999,998</td>
<td>1 in 292,201,338</td>
<td>0.00000000034</td>
<td></td>
<td>$1.54</td>
</tr>
<tr>
<td>5 white</td>
<td>$1,000,000</td>
<td>$999,998</td>
<td>1 in 11,688,053.52</td>
<td>0.00000000856</td>
<td></td>
<td>$0.09</td>
</tr>
<tr>
<td>4 white and red</td>
<td>$50,000</td>
<td>$49,998</td>
<td>1 in 913,129.18</td>
<td>0.0000010951</td>
<td></td>
<td>$0.05</td>
</tr>
<tr>
<td>4 white</td>
<td>$100</td>
<td>$98</td>
<td>1 in 36,525.17</td>
<td>0.0000273784</td>
<td></td>
<td>$0.00</td>
</tr>
<tr>
<td>3 white and red</td>
<td>$100</td>
<td>$98</td>
<td>1 in 14,494.11</td>
<td>0.0000689935</td>
<td></td>
<td>$0.01</td>
</tr>
<tr>
<td>3 white</td>
<td>$7</td>
<td>$5</td>
<td>1 in 579.76</td>
<td>0.0017248517</td>
<td></td>
<td>$0.01</td>
</tr>
<tr>
<td>2 white and red</td>
<td>$7</td>
<td>$5</td>
<td>1 in 701.33</td>
<td>0.0014258623</td>
<td></td>
<td>$0.01</td>
</tr>
<tr>
<td>1 white and red</td>
<td>$4</td>
<td>$2</td>
<td>1 in 91.98</td>
<td>0.0108719287</td>
<td></td>
<td>$0.02</td>
</tr>
<tr>
<td>Red</td>
<td>$4</td>
<td>$2</td>
<td>1 in 38.32</td>
<td>0.0260960334</td>
<td></td>
<td>$0.05</td>
</tr>
<tr>
<td>Nothing</td>
<td>$0</td>
<td>-$2</td>
<td>1 in 1.04</td>
<td>0.9597837679</td>
<td></td>
<td>-$1.92</td>
</tr>
</tbody>
</table>

**Expected Value:** -$0.14

*Source: Business Insider calculations with odds from powerball.com*
• Flip a fair coin until it lands tails
• If we flipped the coin \( n \) times, you get \( 2^n \)
• How much would you be willing to pay to participate?

\[
\mathbb{E}[2^n] = \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 4 + \frac{1}{8} \cdot 8 + \ldots = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot 2^n = \infty
\]

\[
\mathbb{E}[\log(2^n)] = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \log(2^n) = \log(2) \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot n = 2 \log(2) \approx 0.60
\]
von Neumann and Morgestern
A simple lottery is a tuple $L = (p_1, x_1; p_2, x_2; \ldots p_n, x_n)$

- Monetary prizes $x_1, \ldots, x_n \in X \subseteq \mathbb{R}$
- Probability distribution $(p_1, \ldots, p_n)$, $p_i$ is the probability of $x_i$

Let $\mathcal{L}$ denote the set of simple lotteries

Example: $L = (10, 0.2; 5, 0.1; 0, 0.3; -5, 0.4)$
Simple lotteries given a **fixed** set of prizes $x_1, \ldots, x_n$ correspond to points in the $n$-dimensional simplex

$$\Delta^n = \left\{ (p_1, \ldots, p_n) \in \mathbb{R}^n \mid 0 \leq p_i \leq 1 \; \& \; p_1 + \ldots + p_n = 1 \right\}$$
• Simple lotteries given a **fixed** set of prizes $x_1, \ldots, x_n$ correspond to points in the $n$-dimensional simplex

$$
\Delta^n = \left\{ (p_1, \ldots, p_n) \in \mathbb{R}^n \mid 0 \leq p_i \leq 1 \quad \& \quad p_1 + \ldots + p_n = 1 \right\}
$$
lottery mixtures

• For $0 \leq \alpha \leq 1$ and lotteries $L = (p_1, x_1; p_2, x_2; \ldots p_n, x_n)$ and $M = (q_1, x_1; q_2, x_2; \ldots q_n, x_n)$ with the same set of prizes, define

$$\alpha L \oplus (1 - \alpha) M = (r_1, x_1; r_2, x_2; \ldots r_n, x_n)$$

where

$$r_k = \alpha p_k + (1 - \alpha) q_k$$

• Example: $L = (0.5, 10; 0.5, 5)$, $M = (0.8, 10; 0.2, 5)$, $\alpha = 0.6$
• With a fixed set of prizes $x_1, \ldots, x_n$, mixtures between lotteries correspond to points in the line segment between them.

• The mixture weights determine the location within the segment.

```
(0,0,1)

M

0.2L \oplus 0.8M

0.8\|L-M\|

L

(1,0,0)

(0,1,0)
```
• Also possible to mix lotteries with different prizes

• **Example:** \( L = (0.5, 10; 0.5, 5), M = (0.8, 20; 0.2, 5), \alpha = 0.6 \)

\[ \alpha L \oplus (1 - \alpha)M = (0.3, 10; 0.32, 20; 0.38, 5) \]
• Reported preferences $\succ$ on $\mathcal{L}$

• A utility function $U : \mathcal{L} \to \mathbb{R}$ for $\succ$ is an expected utility function if it can be written as

$$U(L) = \sum_{k=1}^{n} p_i u(x_i)$$

for some function $u : \mathbb{R} \to \mathbb{R}$

• If you think of the prizes as a random variable $\mathbf{x}$, then

$$U(L) = \mathbb{E}_L [ u(\mathbf{x}) ]$$

• The function $u$ is called a Bernoulli utility function
expected utility axioms

• **Axiom 1**: (Preference order) $\succ$ is a asymmetric and negatively transitive

• **Axiom 2**: (Continuity) For all simple lotteries $L, M, N \in \mathcal{L}$, if $L \succ M \succ N$ then there exist $\alpha, \beta \in (0, 1)$ such that

$$\alpha L \oplus (1 - \alpha)N \succ M \succ \beta N \oplus (1 - \beta)N$$

• **Axiom 3**: (Independence) For all lotteries $L, M, N \in \mathcal{L}$ and $\alpha \in (0, 1]$, if $L \succ M$, then

$$\alpha L \oplus (1 - \alpha)N \succ \alpha M \oplus (1 - \alpha)N$$
• The continuity axiom can be thought of as requiring that strict preference is preserved by sufficiently small perturbations in the probabilities
  
  – If \( L \succ M \), then so are lotteries which are close enough to \( L \) (hatched area)
  – This includes \( \alpha L \oplus (1 - \alpha)N \) with \( \alpha \) close enough to 1
• If $L$ is preferred to $M$, then a mixture of $L$ with $N$ is also preferred to a mixture of $M$ with $N$ using the same mixing weights.

• Independence gives the expected-utility structure.

• Similar to the independent-factors requirement from previous notes (expected utility is a form of additive separability).
• How do you rank the following lotteries?

$L\bigcirc$
\begin{align*}
  0.4 &\bullet 600 \\
  0.6 &\bullet 400
\end{align*}

$M\bigcirc$
\begin{align*}
  0.8 &\bullet 1500 \\
  0.2 &\bullet -100
\end{align*}

• How do you rank the following lotteries?

$L'\bigcirc$
\begin{align*}
  0.1 &\bullet 1500 \\
  0.2 &\bullet 600 \\
  0.3 &\bullet 500 \\
  0.4 &\bullet 400
\end{align*}

$M'\bigcirc$
\begin{align*}
  0.5 &\bullet 1500 \\
  0.3 &\bullet 500 \\
  0.1 &\bullet 400 \\
  0.1 &\bullet -100
\end{align*}

• Independence says that if you prefer $L$ to $M$, then you also prefer $L'$ to $M'$

• Note that $L' = 0.5L \oplus 0.5N$ and $M' = 0.5M \oplus 0.5N$, for some lottery $N$ (which lottery?)
• How do you rank the following lotteries?

\[
L_1 \circ \quad 1 \quad \bullet \quad 1,000,000$
\]

\[
M_1 \circ \quad 0.1 \quad \bullet \quad 5,000,000$
\]

\[
M_1 \circ \quad 0.89 \quad \bullet \quad 1,000,000$
\]

\[
M_1 \circ \quad 0.01 \quad \bullet \quad 0$
\]

• Many people report \( L_1 \succ M_1 \) and \( M_2 \succ L_2 \).
• Note that we can write

\[ L_1 \succ M_1 \text{ if and only if } L_2 \succ M_2 \] (why?)
von neumann-morgenstern theorem

Theorem:

(a) A binary relation $\succ$ over $\mathcal{L}$ has an expected utility representation if and only if it satisfies axioms 1–3

(b) If $U$ and $V$ are expected utility representations of $\succ$, then there exist constants $a, b \in \mathbb{R}$, $a > 0$, such that $U(\cdot) = a \cdot V(\cdot) + b$
proof of necessity

• Suppose $U$ is an expected utility representation of $\succ$
• Axiom 1 follows from the same arguments as before
• For $0 \leq \alpha \leq 1$ and lotteries $L = (p_1, x_1; p_2, x_2; \ldots p_n, x_n)$ and $M = (q_1, x_1; q_2, x_2; \ldots q_n, x_n)$ note that

$$V(\alpha L \oplus (1 - \alpha)M) = \sum_{i=1}^{n} \left[ \alpha p_i + (1 - \alpha)q_i \right] \cdot u(x_i)$$

$$= \sum_{i=1}^{n} \left[ \alpha p_i u(x_i) + (1 - \alpha)q_i u(x_i) \right]$$

$$= \alpha \sum_{i=1}^{n} p_i u(x_i) + (1 - \alpha) \sum_{i=1}^{n} q_i u(x_i)$$

$$= \alpha V(L) + (1 - \alpha)V(M)$$

• From here, it is straightforward to show that $\succ$ satisfies axioms 2 & 3
• Fix the set of prizes so that lotteries can be thought of as vectors in $\Delta^n$
• The following proposition that, under axioms 1–3, preferences are preserved under translations
• This means that the indifference curves are parallel lines

**Proposition:** Given lotteries $L, M \in \Delta^n$, and a vector $N \in \mathbb{R}^n$, if $L + N$ and $M + N$ are also lotteries and $\succ$ satisfies axioms 1–3, then

$$L \succ M \iff (L + N) \succ (M + N)$$
Proof sketch:

- If \((L + N)\) and \((M + N)\) are lotteries, then so are \(A\) and \(B\)
- \(A = 0.5M \oplus 0.5(L + N)\) and \(A = 0.5L \oplus 0.5(M + N)\)
- Since \(A = 0.5M \oplus 0.5(L + N)\), independence says that if \(L \succ M\) then \(B \succ A\)
- Since \(A = 0.5L \oplus 0.5(M + N)\), independence says that if \(B \succ A\) then \((L + N) \succ (M + N)\)
risk aversion
risk attitudes

• For the rest of these slides, suppose \( u \) is strictly increasing (our decision maker always prefers more money) and twice continuously differentiable

• Risk-neutral decision maker – \( \mathbb{E}[u(x)] = u(\mathbb{E}[x]) \) for every random variable \( x \)

• Risk-averse decision maker – \( \mathbb{E}[u(x)] \leq u(\mathbb{E}[x]) \) for every r.v. \( x \)

• Risk-loving decision maker – \( \mathbb{E}[u(x)] \geq u(\mathbb{E}[x]) \) for every r.v. \( x \)
• A set is convex if it contains all the line-segments between its points
• A function is concave if its hypograph is a convex set
• Risk aversion is equivalent to $u$ being concave
Definition: Given $u$, the certainty equivalent of a lottery $x$ is the guaranteed amount of money that an individual with Bernoulli utility function $u$ would view as equally desirable as $x$, i.e.,

$$CE_u(x) = u^{-1}(\mathbb{E}[u(x)])$$

- Risk-neutral decision maker – $CE(L) = \mathbb{E}[x]$ for every r.v. $x$
- Risk-averse decision maker – $CE(L) \leq \mathbb{E}[x]$ for every r.v. $x$
- Risk-loving decision maker – $CE(L) \geq \mathbb{E}[x]$ for every r.v. $x$
**Definition:** The arrow-pratt coefficient of absolute risk aversion of $u$ at $x$ is

$$A_u(x) = -\frac{u''(x)}{u'(x)}$$

- Constant absolute risk aversion (CARA)

$$u(x) = -\exp(-\alpha x)$$

- Indeed $u'(x) = \alpha u(x)$ and $u''(x) = \alpha^2 u(x) \Rightarrow A_u(x) = \alpha$
Given any two strictly increasing Bernoulli utility functions $u$ and $v$, the following are equivalent

(a) $A_u(x) \geq A_v(x)$ for all $x$

(b) $CE_u(x) \leq CE_v(x)$ for all $x$

(c) There exists a strictly increasing concave function $g$ such that $u = g \circ v$

In that case, we say that $v$ is (weakly) more risk averse than $u$
proof sketch

• There always exist a strictly increasing and twice continuously differentiable function \( g \) such that \( v = g \circ u \) (why?)

• By the chain-rule of differential calculus

\[
\begin{align*}
    v'(x) &= g'(u(x))u'(x) \\
    v''(x) &= g'(u(x))u''(x) + g''(u(x))(u'(x))^2
\end{align*}
\]

• If \( g \) is concave, then \( g'' < 0 \) and thus

\[
A_v(x) = -\frac{v''(x)}{v'(x)} = -\frac{g'(u(x))u''(x) + g''(u(x))(u'(x))^2}{g'(u(x))u'(x)}
\]

\[
= A_u(x) - \frac{g''(u(x))u'(x)}{g'(u(x))} \geq A_u(x)
\]
• If $g$ is concave, then Jensen’s inequality implies that

$$
\begin{align*}
v(CE_v(x)) &= E[v(x)] = E[g(u(x))] \\
&\leq g(E[u(x)]) = g(u(CE_u(x))) = v(CE_u(x))
\end{align*}
$$

• Since $v$ is strictly increasing, this implies that

$$CE_v(x) \leq CE_u(x)$$
optimal portfolios
• An expected utility maximizer with initial wealth $\omega$ must decide a quantity $\alpha$ to invest on a risky asset

• The asset has a random gross return of $z$ per dollar invested

• The final wealth of the investor will be $w - \alpha + \alpha z$

• The optimal investment is the solution to the program

$$\max_{\alpha} \mathbb{E} \left[ u(w + \alpha(z - 1)) \right]$$

s.t. $0 \leq \alpha \leq w$

• Let $\alpha^*$ denote this solution
a risky asset

**Proposition:** A risk averse agent will always invest a positive amount on assets with positive expected return, i.e., if $\mathbb{E}[z] > 1$ then $\alpha^* > 0$

**Proof:**

- Let $U(\alpha)$ denote the agent’s expected utility

$$U'(\alpha) = \mathbb{E} [(z - 1)u'(w + \alpha(z - 1))]$$

- If $\mathbb{E}[z] > 1$, then $U$ is strictly increasing at 0 because

$$U'(0) = \mathbb{E} [(z - 1)u'(w)] = u'(w)(\mathbb{E}[z] - 1) > 0$$
• Suppose there are two assets with i.i.d. returns $z_1$ and $z_2$

• The investor chooses investments $\alpha_1, \alpha_2 \geq 0$ with $\alpha_1 + \alpha_2 \leq q$

• Let $U(\alpha_1, \alpha_2)$ denote the investor’s expected utility

$$U(\alpha_1, \alpha_2) = \mathbb{E} \left[ u(w + \alpha_1(z_1 - 1) + \alpha_2(z_2 - 1)) \right]$$

**Proposition:** A risk averse agent will always diversify among risky i.i.d. assets with positive returns, i.e., if $\mathbb{E}[z_i] > 1$ and $\mathbb{V}[z_i] > 0$, then $\alpha_1^* > 0$ and $\alpha_2^* > 0$. 
• We already know that the optimal portfolio cannot be \((0, 0)\) (why?)

• For any portfolio without diversification \((\alpha^0, 0)\) we have that

\[
U(\alpha^0, 0) = \frac{1}{2} \mathbb{E} \left[ u(w + \alpha^0(z_1 - 1)) \right] + \frac{1}{2} \mathbb{E} \left[ u(w + \alpha^0(z_2 - 1)) \right]
\]

\[
= \mathbb{E} \left[ \frac{1}{2} u(w + \alpha^0(z_1 - 1)) + \frac{1}{2} u(w + \alpha^0(z_2 - 1)) \right]
\]

\[
< \mathbb{E} \left[ u \left( \frac{1}{2} (w + \alpha^0(z_1 - 1)) + \frac{1}{2} (w + \alpha^0(z_2 - 1)) \right) \right]
\]

\[
= \mathbb{E} \left[ u \left( w + \frac{1}{2} \alpha^0(z_1 - 1) + \frac{1}{2} \alpha^0(z_2 - 1) \right) \right]
\]

\[
= U \left( \frac{1}{2} \alpha^0, \frac{1}{2} \alpha^0 \right)
\]
comparing distributions
cumulative distribution functions

- The cumulative distribution functions (c.d.f.) of a random variable $x$ is the function $F : \mathbb{R} \to [0, 1]$ given by

$$F(\xi) = \Pr(x \leq \xi)$$

- C.d.f.s are non-decreasing, left-continuous, satisfy $\lim_{\xi \to -\infty} F(\xi) = 0$ and $\lim_{\xi \to \infty} F(\xi) = 1$
• Consider random variables $x$ and $y$ with c.d.f.s $F$ and $G$
• That is $F(\xi) = \Pr(x \leq \xi)$ and $G(\xi) = \Pr(y \leq \xi)$

• When can we say that $x$ is “greater” than $y$?
  - $\mathbb{E}[x] > \mathbb{E}[y]$ is probably not enough
  - $\min\{\text{support}(x)\} > \max\{\text{support}(y)\}$ is probably too much

• When can we say that $x$ is “riskier” than $y$?
  - $\mathbb{V}[x] > \mathbb{V}[y]$ is probably not enough
first-order stochastic dominance

• Say that $F$ first-order stochastically dominates $G$ if every expected utility maximizer with monotone preferences would choose $x$ over $y$

**Definition:** Say that $F \succ_{\text{FOSD}} G$ if for every non-decreasing function $u : \mathbb{R} \to \mathbb{R}$ we have that $\mathbb{E}[u(x)] \geq \mathbb{E}[u(y)]$

• First-order stochastic dominance can be characterized in terms of distribution functions

• The following proposition asserts that $x \succ_{\text{FOSD}} y$ if for every number $\xi$, $y$ taking a value smaller than $\xi$ is more likely than $x$ taking a value smaller than $\xi$

**Proposition:** $x \succ_{\text{FOSD}} y$ if and only if $F(\xi) \leq G(\xi)$
first order stochastic dominance

$G(\xi), F(\xi)$

$F \succ_{\text{FOSD}} G$
proof sketch

• Suppose $F(\xi) > G(\xi)$ for some $\xi$
  
  - Let $u : \mathbb{R} \to \mathbb{R}$ be the Bernoulli utility function
    
    $$u(x) = \begin{cases} 
    1 & \text{if } x > \xi \\
    0 & \text{otherwise} 
    \end{cases}$$

  - Then $\mathbb{E}[u(x)] = 1 - F(\xi) < 1 - G(\xi) = \mathbb{E}[u(y)]$

• Suppose $F(\xi) \leq G(\xi)$ for all $\xi$ and $u$, $F$ and $G$ are differentiable
  
  - Integrating by parts:
    
    $$\mathbb{E}[u(x)] = - \int_{-\infty}^{\infty} u'(\xi) F(\xi) \, d\xi$$

  - Therefore
    
    $$\mathbb{E}[u(x)] - \mathbb{E}[u(y)] = - \int_{-\infty}^{\infty} u'(\xi) (F(\xi) - G(\xi)) \, d\xi \geq 0$$
second order stochastic dominance

• First-order stochastic dominance is a very incomplete ranking
• More comparisons if we further restrict the set of utility functions
• Say that $F$ second-order stochastically dominates $G$ if every expected utility maximizer with monotone and concave preferences would choose $x$ over $y$

**Definition:** Say that $F \succ_{\text{SOSD}} G$ if for every non-decreasing and concave function $u : \mathbb{R} \to \mathbb{R}$ we have that $\mathbb{E}[u(x)] \geq \mathbb{E}[u(y)]$

• Since concavity is a measure of risk-aversion, second-order stochastic dominance helps us to rank distributions by how much risk they involve
mean preserving spreads

• Say that \( y \) is a mean preserving spread of \( x \) if we can write

\[
y = x + \varepsilon
\]

where \( \mathbb{E}[\varepsilon|x] = 0 \)

• That is, \( y \) equals \( x \) plus “noise”

**Proposition:** The following are equivalent

(a) \( F \succ_{\text{SOSD}} G \)

(b) There exist random variables \( x \) and \( y \) with c.d.f.s \( F \) and \( G \), resp., such that \( y \) is a mean preserving spread of \( x \)

(c) For every number \( \xi \)

\[
\int_{-\infty}^{\xi} F(x) \, dx \leq \int_{-\infty}^{\xi} G(y) \, dy
\]
second order stochastic dominance

\[ F \succ_{\text{SOSD}} G \]