Utility Representations

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Cornell University · Decision Theory · Spring 2017

utility representations

- Can we represent preferences numerically?

> **Definition:** A (utility) function $u : X \to \mathbb{R}$ is a utility representation of $\succ$ if for every $x, y \in X$
>
> $$x \succ y \quad \Leftrightarrow \quad u(x)\ u(y)$$

- A utility representation makes it easier to compare choices
  - Asparagus is a 5 and kale is a 1: obviously I prefer asparagus to kale!
- A utility representation is easier to think about than an ordering
- It's also typically easier to find an optimal choice maximizing a utility function (e.g., using calculus)

- If u represents $\succ$, then we must have $x \not\succ y$ if and only if $u(x) \leq u(y)$
- Thus, $x \sim y$ iff $u(x) = u(y)$

- **Example:** $X = \{x, y, z\}$, $x \succ_{po} z$, and y is incomparable to x and z
  - $\succ_{po}$ has no utility representation
  - Since $x \sim y$ and $y \sim z$, we would need $u(x) = u(y)$ and $u(y) = u(z)$
  - Since $x \succ z$ we would need $u(x) > u(z)$
  - Hnce we would need $u(x) = u(z)$ and $u(x) > u(z)$ ▼

- So when does an ordering have a utility representation?

Proposition: If $\succ$ has a utility representation, then $\succ$ is a preference order

Proof:

- Suppose that $\succ$ has a utility representation u
- We must show that $\succ$ is asymmetric and negatively transitive
  - If $x \succ y$, then $u(x) > u(y)$, so $u(y) \not> u(x)$, so $x \not\succ y$
  - If $x \not\succ y$ and $y \not\succ z$, then $u(x) \leq u(y)$ and $u(y) \leq u(z)$, so $u(x) \leq u(z)$, so $x \not\succ z$

Theorem: Given a finite set X, a binary relation $\succ$ on X is a preference order if and only if $\succ$ has a utility representation

Proof:

- Recall that if $\succ$ is a preference order on X, then we can partition the elements of X into "indifference classes" $X_1, \ldots, X_k$ such that "$X_1 \succ X_2 \succ \ldots \succ X_k$"
- Thus, we can define u so that $u(x) = k$ for all $x \in X_1$, $u(x) = k - 1$ for all $x \in X_2$, ..., $u(x) = 1$ for all $x \in X_k$
- Here is a more formal proof by induction...

proof for finite case

- Suppose that $X = \{x_1, \ldots, x_n\}$
- We show that if $\succ$ is a preference order on X then $\succ$ has a utility representation, by induction on n, the number of elements in X
- if $n = 1$, then just take $u(x) = 1$ and we are done

- Suppose the result holds if X has cardinality $n - 1$ (i.e., $n - 1$ elements)
- If $\succ$ is a preference order on X, then it also a preference order on $X' = X \setminus \{x_n\} = \{x_1, \ldots, x_{n-1}\}$
  - This needs to be checked!
- By the induction hypothesis, since X' has $n - 1$ elements, there is a utility function $u : X' \to R$ such that $u(x) > u(y)$ iff $x \succ y$ for all $x, y \in X'$
- How do we extend u to x_n?

- Before proceeding with the proof, recall a useful property that we will use a few times today
- In words, if two alternatives x and y are not comparable, then every other alternative z ranks relative to x the same way it ranks relative to y

Lemma: If $\succ$ is a strict preference and $x \sim y$, then

(a) $x \succ z$ iff $y \succ z$

(a) $z \succ x$ iff $z \succ y$

Proof:

- By NT, if $x \not\succ z$, then $y \not\succ x \not\succ z$ and thus $y \not\succ z$
- By NT, if $z \not\succ x$, then $z \not\succ x \not\succ y$ and thus $z \not\succ y$

There are four cases to consider

1. If $x_n \sim y$ for some $y \in X'$, set $u(x_n) = u(y)$
   - By the lemma, $x \succ z$ iff $y \succ z$ iff $u(x_n) = u(y) > u(z)$
2. If $x_n \succ y$ for all $y \in X'$, set $u(x_n) = 1 + \max_{y \in X'} u(y)$
   - For every $z \neq x_n$, $x_n \succ z$ and $u(x_n) > u(z)$
3. If $y \succ x_n$ for all $y \in X'$, set $u(x_n) = \min_{y \in X'} u(y) - 1$
   - For every $z \neq x_n$, $z \succ x_n$ and $u(z) > u(n_n)$
4. If none of the previous cases apply, there exist y and y' in X' such that $y \succ x_n \succ y'$ and we can set $u(x_n) = 0.5 \min_{y \succ x} u(y) + 0.5 \max_{x \succ y} u(y)$
   - By transitivity of $\succ$, $\max_{x \succ y} u(y) < u(x_n) < \min_{y \succ x} u(y)$ (why?)
   - hence $x \succ z$ iff $u(z) \leq \max_{x \succ y} u(y) < u(x)$

- Is utility uniquely defined? only up to monotone transformations

> **Proposition:** If u represents $\succ$ and $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then $f \circ u$ also represents $\succ$

- **Proof:** $x \succ y$ iff $u(x) > u(y)$ iff $f(u(x)) > f(u(y))$
- Be careful with interpretation of ordinal utility:
  - Decreasing marginal utility?
  - Interpersonal comparisons?

Theorem: Given a countable set X, a binary relation $\succ$ on X is a preference order if and only if $\succ$ has a utility representation

Proof:

- Since X is countable we can label its elements $X = \{x_1, x_2, x_3, \ldots\}$
- Let $W(x) = \{y \in X \mid x \succ y\}$ be the set of alternatives that are worse than x
- Let $N(x) = \{n \mid x_j n \in W(x)\}$ be the set of labels of such alternatives
- Define $u : X \to \mathbb{R}$ by

$$u(x) = \sum_{n \in N(x)} \left(\frac{1}{2}\right)^n$$

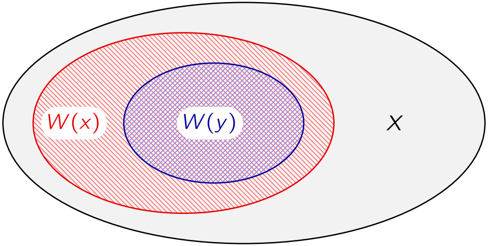
  with the convention that sum over the empty set equals 0
- We need to show that u represents $\succ$

- Suppose $x \succ y$
  - By transitivity, if $y \succ z$ then $x \succ z$, and thus $W(y) \subseteq W(x)$
  - By asymmetry, $y \not\succ y$, and thus $W(y) \subsetneq W(x)$
  - Therefore $N(y) \subseteq N(x)$ and thus $u(x) > u(y)$

- Suppose $u(x) > u(y)$
  - From previous point, $y \not\succ x$
  - If $x \sim y$, by our lemma, $W(x) = W(y)$
  - This would imply $u(x) = u(y)$ contradicting our assumption $\blacktriangledown$
  - Hence $y \not\sim y$ and $y \not\succ x$, and thus $x \succ y$

If X is countably infinite, then the function $\Pr : X \to [0,1]$ given by $\Pr(x_n) = 2^{-n}$ is a probability function on X with full support and

$$u(x) = \sum_{y \in W(x)} \Pr(y) = \Pr\left(W(x)\right)$$

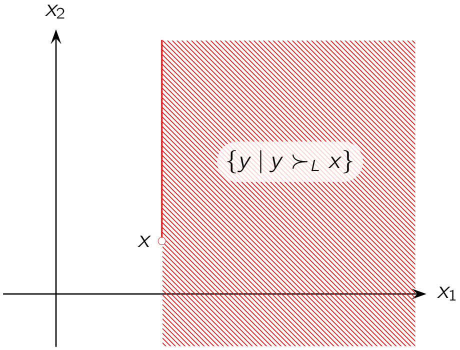


$$x \succ y \quad \Leftrightarrow \quad W(x) \supsetneq W(y) \quad \Leftrightarrow \quad \Pr\left(W(x)\right) > \Pr\left(W(y)\right)$$

lexicographic preferences

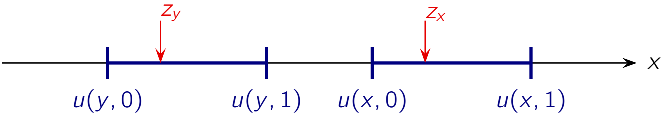
- Do all preference orders admit a utility representation? No
- **Example:** $X = \mathbb{R}^2$ and $\succ_L$ is the lexicographic or dictionary order given by

$$x \succ_L y \quad \Leftrightarrow \quad \begin{cases} x_1 > y_1 \\ \text{or} \\ x_1 = y_1 \text{ and } x_2 > y_2 \end{cases}$$



- $\succ_L$ is asymmetric
  - If $x \succ_L y$ then $x_1 > y_1$ or $[x_1 = y_1$ and $x_2 > y_2]$
  - If $x_1 > y_1$ then neither $y_1 > x_1$ nor $y_1 = x_1$, and thus $y \not\succ_L x$
  - If $[x_1 = y_1$ and $x_2 > y_2]$ then neither $y_1 > x_1$ nor $[y_1 = x_1$ and $y_2 > x_2]$, and thus $y \not\succ_L x$
- $\succ_L$ is negatively transitive
  - If $x \neq y$ and $x \not\succ_L y$, then $y \succ_L x$
    - $x \not\succ_L y$ implies either $x_1 < y_1$ or $[x_1 = y_1$ and $x_2 \leq y_2]$
    - If $x \neq$y, this implies either $y_1 > x_1$ or $[y_1 = x_1$ and $y_2 > x_2]$
  - If $x \not\succ y \not\succ z$ and $x = y$, or $y = z$, or $x = z$, then $x \not\succ z$ (why?)
  - Suppose $x \not\succ y \not\succ z$, $x \neq y$, $x \neq z$ and $y \neq z$
    - Then $z \succ y \succ x$
    - If $z_1 > y_1 \geq x_1$ then $z \succ x$ and thus $x \not\succ z$
    - If $z_1 \geq y_1 > x_1$ then $z \succ x$ and thus $x \not\succ z$
    - If not, then $z_1 = y_1 = x_1$ and $z_2 > y_2 > x_2$, then $z \succ x$ and thus $x \not\succ z$

- $\succ_L$ does not admit a utility representation
  - Suppose $u : X \to \mathbb{R}$ represents $\succ_L$
  - For every $x, y \in \mathbb{R}$ if $x > y$ then $(x, 1) \succ_L (x, 0) \succ_L (y, 1)$
  - Hence, if $x > y$ then we must have $u(x, 1) > u(x, 0) > u(y, 1)$
  - Therefore, the intervals $\{ [u(x, 0), u(x, 1)] \mid x \in \mathbb{R}\}$ are all disjoint
  - Moreover, each of these intervals contains a rational number $z_x \in [u(x, 0), u(x, 1)] \cap \mathbb{Q}$



  - Hence, we have constructed a one-to-one function from $\mathbb{R}$ to $\mathbb{Q}$, which is not possible because $\mathbb{Q}$ countable ▼
  - Hence, there cannot exist a utility representation for $\succ_L$

- Trying to "fit" $(X, \succ)$ into $(\mathbb{R}, >)$
- The properties of $(X, \succ)$ must be compatible with those of $(\mathbb{R}, >)$
  - $>$ is NT and A $\Rightarrow \succ$ must be NT and A
  - For every two reals $x, y \in \mathbb{R}$, if $x > y$, there exists a rational number $z \in \mathbb{Q}$ such that $x > z > y$

Definition: A set $Z \subseteq X$ is order-dense with respect to $\succ$ if for every $x, y \in X \setminus Z$ such that $x \succ y$, there exists some $z \in Z$ such that $x \succ z \succ y$

Theorem: Given an arbitrary set X and a binary relation $\succ$ on X, $\succ$ has a utility representation if and only if

(a) $\succ$ is a preference order

(b) X has a countable order-dense subset with respect to $\succ$

- This result also covers the finite and countable cases (why?)
- A similar construction to the countable case works
  - Enumerate $Z = \{z_1, z_2, \ldots\}$
  - Define $N(x) = \{n \mid x \succ z_n\}$ and $u(x) = \sum_{n \in N(x)} 2^{-n}$

- Say that $\succ$ is continuous if whenever $x \succ y$ and $x_n \longrightarrow x$, there exists some N such that for $n \geq N$ we have $x_n \succ x$

Proposition: If $\succ$ is a continuous preference order, then there exists a continuous function $u : X \rightarrow \mathbb{R}$ that represents $\succ$

- Partial orders are transitive and asymmetric, but indifference (non-comparability) may not be transitive
- They may fail to have utility representations
- $u : X \to \mathbb{R}$ is a partial utility representation of $\succ$ if

$$x \succ y \qquad \Rightarrow \qquad u(x) > u(y)$$

> **Theorem:** If X is countable, and $\succ$ is transitive and asymmetric, then $\succ$ has a partial utility representation

- Same construction and proof as before works

- Often use models with more structure
- Suppose $X = X_1 \times \ldots \times X_n$ is a product space with n factors
- Typical elements $x = (x_1, \ldots, x_n)$

Definition: A utility function $u : X \to \mathbb{R}$ is additive separable if there exist functions $u_i : X_i \to \mathbb{R}$ such that

$$u(x) = u_1(x_1) + \ldots + u_n(x_n)$$

examples

- Cobb-Douglas utility from consumption bundles

$$u(x, y) = \alpha \log(x) + \beta \log(y)$$

- Expected utility from prizes $(x_1, \dots, x_n)$ with probabilities $(p_1, \dots, p_n)$

$$U(L) = \sum_{i=1}^{n} p_i u(x_i)$$

- Discounted utility from consumption stream $c = (c_0, \dots, c_T)$

$$U(c) = \sum_{t=0}^{T} \delta^t u(c_t)$$

- Preferences over some products with multiple features (health insurance)
- Foundations of partial equilibrium

- Given two alternative $x, y \in X$ and a set of indices $I \subseteq \{1, \dots, n\}$, let $(x_I, y_{-I}) \in X$ denote the alternative $z \in X$ given by

$$z_i = \begin{cases} x_i \text{ if } i \in I \\ y_i \text{ if } i \notin I \end{cases}$$

- $\succ$ satisfies independent factors if for all $x, y, w, z \in X$ and $I \subseteq \{1, \dots, n\}$

$$(x_I, w_{-I}) \succ (y_I, w_{-I}) \qquad \Leftrightarrow \qquad (x_I, z_{-I}) \succ (y_I, z_{-I})$$

- Factor i is essential with respect to $\succ$ if there exist $x, y, z \in X$ such that

$$(x_i, z_{-i}) \succ (y_i, z_{-i})$$

Theorem: If $\succ$ is a continuous preference order on $X = X_1 \times \ldots \times X_n$ with independent factors and there are at least three independent factors then

(a) $\succ$ has an additive separable utility representation u

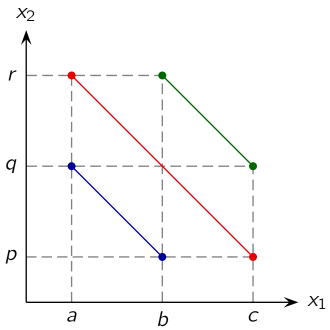
(b) The corresponding u_i functions are continuous

(c) If v is another additive separable utility representation, then there exist $a, b \in \mathbb{R}$ with $a > 0$ such that $b(\cdot) = au(\cdot) + b$

a necessary condition

Proposition: If $X = X_1 \times X_2$ and $\succ$ has an additive separable utility representation, then for all $a, b, c \in X_1$ and $p, q, r \in X_2$

$$\big[(a, q) \sim (b, p) \;\&\; (c, q) \sim (b, r)\big] \quad \Rightarrow \quad (a, r) \sim (c, p)$$



Condorcet paradox

- Is there a natural way of deriving social preferences from individual preferences?
- **Example:** Condorcet Paradox
  - Anna, Bob and Charlie's preferences are given by

$$x \succ_A y \succ_A z \qquad y \succ_B z \succ_B x \qquad z \succ_C x \succ_C y$$

  - One could construct social preferences $\succ$ by simple majority voting

$$x \succ y \qquad y \succ z \qquad z \succ x$$

  - This would violate transitivity!