Von Neumann – Morgenstern Expected Utility

I. Introduction, Definitions, and Applications

Decision Theory
Spring 2014

Origins

Blaise Pascal, 1623 – 1662

- Early inventor of the mechanical calculator
- Invented Pascal's Triangle
- Invented expected utility, hedging strategies, and a cynic's argument for faith in God all at once.

Pascal's Wager

	God exists	God does not exist
live as if he does	$-C+\infty$	$-C$
live for yourself	$U-\infty$	U

Origins

Daniel Bernoulli

- Mechanics
- Hydrodynamics — Kinetic Theory of Gases
- Bernoulli's Principle

The St. Petersburg Paradox

A coin is tossed until a tails comes up. How much would you pay for a lottery ticket that paid off 2^n dollars if the first tails appears on the n'th flip?

The St. Petersburg Paradox

Average payoff from paying c:

$$\begin{aligned} E &= \frac{1}{2} \cdot (w - c) + \frac{1}{4} \cdot (w + 2 - c) + \frac{1}{8} \cdot (w + 4 - c) + \cdots \\ &= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + w - c \\ &= \infty \end{aligned}$$

Bernoulli's solution:

$$E = \frac{1}{2} \cdot \log^{+}(w - c) + \frac{1}{4} \cdot \log^{+}(w + 2 - c) + \frac{1}{8} \cdot \log^{+}(w + 4 - c) + \cdots$$

This is finite for all w and c. ($\log^{+}(x) = \log \max\{x, 1\}$.)

The St. Petersburg Paradox

This doesn't solve the problem. Average payoff from paying c:

$$\begin{aligned} E &= \frac{1}{2} \cdot \log^{+}(w + \exp 2^0 - c) + \frac{1}{4} \cdot \log^{+}(w + \exp 2^1 - c) \\ &\quad + \frac{1}{8} \cdot \log^{+}(w + \exp 2^2 - c) + \cdots \\ &= \infty \end{aligned}$$

Solution:

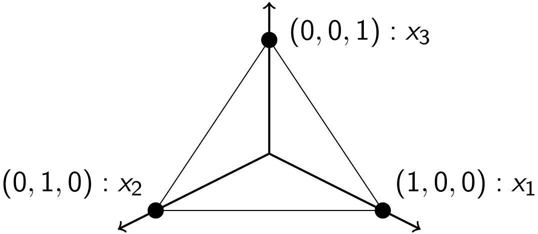
- Utility bounded from above.
- Restrict the set of gambles.

Preferences on Lotteries

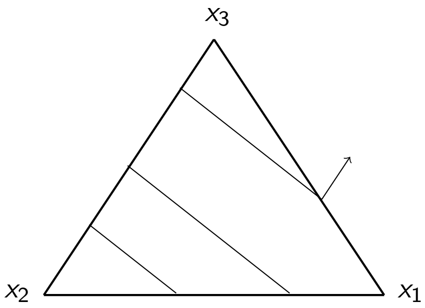
$X = \{x_1, \ldots, x_N\}$ — A finite set of prizes.

$P = \{p_1, \ldots, p_N\}$ — A probability distribution.

p_i is the probability of x_i.



Preferences on Lotteries



For fixed prizes, indifference curves are linear in probabilities.

Lotteries on $\mathbf{R}$

A *simple lottery* is $p = (p_1 : x_1, \dots, p_K : x_K)$ where $x_1, \dots, x_K$ are prizes in $\mathbf{R}$ and $p_1, \dots, p_K$ are probabilities. Let $\mathcal{L}$ denote the set of simple lotteries. Let

$$u : X \to \mathbf{R}$$

$$\text{and } V(p) = \sum_k u(x_k) p(x_k).$$

This is the expectation of the random variable $u(x)$ when the the random variable x is described by the probability distribution p.

Lotteries on **R**

How do we see that this is "linear" in lotteries? For $0 < \alpha < 1$ and lotteries $p = (p_1 : x_1, \ldots, p_K : x_K)$ and $q = (q_1 : y_1, \ldots, q_L : y_L)$, define

$$z_m = \begin{cases} x_m & \text{for } m \leq K, \\ y_{m-K} & \text{for } K < m \leq K + L. \end{cases}$$

and

$$\alpha p \oplus (1 - \alpha) q = (r_1 : z_1, \ldots, r_{K+L} : z_{K+L})$$

where

$$r_m = \begin{cases} \alpha p_m & \text{for } m \leq K, \\ (1 - \alpha) q_{m-K} & \text{for } K < m \leq K + L. \end{cases}$$

Lotteries on $\mathbf{R}$

$$
\begin{aligned}
V(\alpha p \oplus (1-\alpha) q) &= \sum_{m=1}^{M} r_m u(z_m) \\
&= \sum_{m=1}^{K} \alpha p_m u(x_m) + \sum_{m=K+1}^{K+L} (1-\alpha) q_{m-K} u(y_{m-K}) \\
&= \alpha \sum_{k=1}^{K} p_k u(x_k) + (1-\alpha) \sum_{l=1}^{L} q_l u(y_l) \\
&= \alpha V(p) + (1-\alpha) V(q).
\end{aligned}
$$

Attitudes Towards Risk

$$V(p) = E_p\{u\} = \sum_{x \in \mathrm{supp}\, p} u(x)p(x) \qquad \mathcal{P} = \{p \in \mathcal{L} : V(p) < \infty\}.$$

Definition: An individual is risk averse iff for all $p \in \mathcal{P}$, $E_p\{u\} \leq U(E_p\{x\})$. He is risk-loving iff $E_p\{u\} \geq u(E_p\{x\})$.

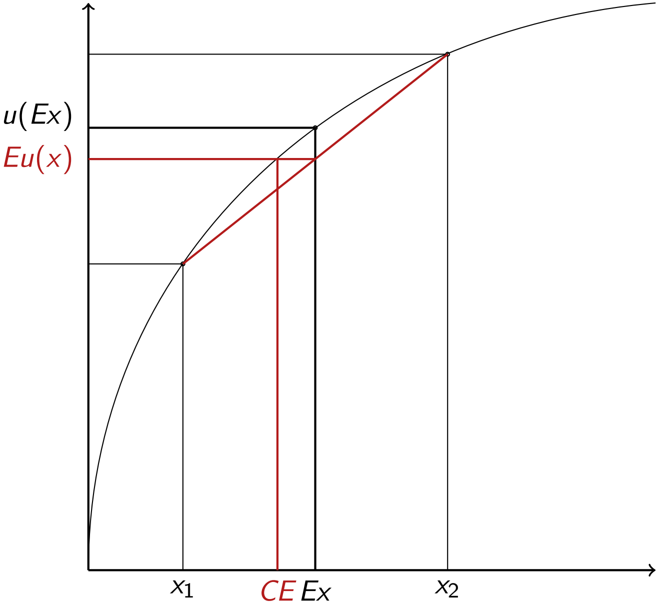
Theorem A: A Decision-Maker is risk averse iff u is concave, and risk-loving iff u is convex.

Definition: The certainty equivalent of a lottery p is the sure-thing amount which is indifferent to p: $CE\{p\} = u^{-1}(V(p))$.

Theorem B: A DM is risk-averse iff for all $p \in \mathcal{P}$, $CE\{p\} \leq E_p\{u\}$.

Proofs are here.

Certainty Equivalents



Measuring Risk Aversion

Definition: The Arrow-Pratt coefficient of absolute risk aversion at x is $r_A(x) = -u''(x)/u'(x)$. The coefficient of relative risk aversion is $r_R(x) = x r_A(x)$.

$$\text{CARA utility: } u(x) = -\exp\{-\alpha x\}.$$

$$\text{CRRA utility: } u(x) = \begin{cases} \frac{1}{\gamma} x^{\gamma}, & \gamma \leq 1, \gamma \neq 0 \\ \log x. \end{cases}$$

When utility is CARA and $\mathcal{P}$ is the set of normal distributions,

$$CE(F) = \mu - \frac{1}{2}\alpha\sigma^2.$$

Comparing Attitudes to Risk

Theorem C: The following are equivalent for two utility functions u_1 and u_2 when $p \in \mathcal{P}$:

1. $u_1 = g \circ u_2$ for some concave g;
2. $CE_1(p) \leq CE_2(p)$ for all $p \in \mathcal{P}$;
3. $r_{A,1}(x) \geq r_{A,2}(x)$ for all $x \in \mathbf{R}$.

How should risk aversion vary with wealth?

$$r_A(x|w) = -u''(x+w)/u'(x+w).$$

How would you expect this to behave as a function of w?

Click for the Proof of Theorem C.

Applications

A risk-averse DM has wealth $w > 1$ and may lose 1 with probability p. He can buy any amount of insurance he wants sat q per unit. His expected utility from buying d dollars of insurance is

$$EU(d) = (1-p)u(w - qd) + pu\big(w - qd - (1-d)\big).$$

Under what conditions will he insure, and for how much of the loss?

Definition: Insurance is actuarially fair, sub-fair, or super-fair if the expected **net payout** per unit, $p - q$, is $= 0$, < 0, or > 0, respectively.

Definition: Full insurance is $d = 1$.

Sub-Fair Insurance

Use derivatives to locate the optimal amount of insurance.

$$EU'(d) = -(1-p)u'(w-qd)q + pu'\big(w - qd - (1-d)\big)(1-q).$$

Suppose $q > p$.

$$EU'(1) = u'(w-q)\big(-(1-p)q + p(1-q)\big) = u'(w-q)(p-q) < 0$$

$$\begin{aligned} EU'(0) &= -(1-p)u'(w)q + pu'(w-1)(1-q) \\ &= pu'(w-1) - \big((1-p)u'(w) + pu'(w-1)\big)q > \cdots p \\ &= p(1-p)\big(u'(w-1) - u'(w)\big) > 0. \end{aligned}$$

Why is $u'(w-1) - u'(w) > 0$?

Fair Insurance

Suppose $q = p$.

$$EU'(1) = u'(w - q)(p - q) = u'(w - q)(p - p)0.$$

Full insurance is optimal.

Optimal Portfolio Choice

A risk-averse DM has initial wealth w. There is a risky asset that pays off z for each dollar invested. z is drawn from a distribution with a probability function p. If he invests in α units of the asset, he gets

$$U(\alpha) = E_p u\big(w + \alpha(z-1)\big)$$

Which is concave in α. The optimal investment α^* solves

$$U'(\alpha^*) = E_p u'\big(w + \alpha(z-1)\big)(z-1) = 0.$$

Optimal Portfolio Choice

He will invest nothing if $E_p\{z\} \leq 1$.

$$U'(0) = E_p u'(w)(z-1) = u'(w)\big(E_p\{z\} - 1\big) \leq 0.$$

If $E_p\{z\} > 1$, then

$$U'(0) = E_p u'(w)(z-1) = u'(w)\big(E_p\{z\} - 1\big) > 0.$$

and so he will hold some positive amount of the asset.

Optimal Portfolio Choice

Theorem: More risk individuals hold less of the risky asset, other things being equal.

Proof: Suppose DM 1 has concave utility u_1, and individual 2 is more risk-averse. Then $u_2 = g \circ u_1$. There is no loss of generality in assuming $g'(u_1) = 1$ at $u_1 = u_1(w)$. For every α,

$$U_2'(\alpha) - U_1'(\alpha) = E_p\big(g'(u_1) - 1\big)u_1'\big(w + \alpha(z-1)\big)(z-1)$$

Now $z < 1$ iff $w + \alpha(z-1) < w$ iff $u_1(w + \alpha(z-1) < u_1(w)$ iff $g'\big(u_1(w + \alpha(z-1))\big) > g'\big(u_1(w)$, so the expression inside the integral is always negative, and so $U_2'(\alpha) < U_1'(\alpha)$ for all α. In particular, when α is optimal for DM 1, $U_2'(\alpha) < 0$, so the optimal α for DM 2 is less.

Comparative Statics

Utility over portfolios depends upon the DM's initial wealth.

$$U(\alpha; w) = E_p u\big(w + \alpha(z-1)\big)$$

The DM's preferences have decreasing absolute risk aversion if whenever $w' > w$, $U(\,\cdot\,; w')$ is less risk-averse then $U(\,\cdot\,; w)$.

Theorem: If the DM has decreasing absolute risk aversion, then α^* increasing in w.

PROOFS

Concavity and Risk Aversion

Definition: A set $C \subset \mathbf{R^k}$ is convex if it contains the line segment connecting any two of its members. function: If $x, y \in C$ and $0 \leq \alpha \leq 1$, $\alpha x + (1 - \alpha) y \in C$.

Definition: A function $f : \mathbf{R^k} \to \mathbf{R}$ is concave iff $\{(x, y) \in \mathbf{R^{k+1}} : y \leq f(x)\}$ is convex.

Jensen's Inequality: A function $f : \mathbf{R^k} \to \mathbf{R}$ is concave if and only if for every N-tuple of numbers $\lambda_1, \ldots, \lambda_N$ that are non-negative numbers and sum to 1, and corresponding $x_1, \ldots, x_N$ are vectors in $\mathbf{R^k}$,

$$\sum_{n=1}^{N} \lambda_n f(x_n) \leq f\left(\sum_{n=1}^{N} \lambda_n x_n\right).$$

Proof: The definition of concavity is Jensen's inequality for $N = 2$. The result for arbitrary N follows from induction.

Concavity and Risk Aversion

Proof of Theorem A: For concave functions this **is** Jensen's inequality. A function f is convex iff $-f$ is concave. Suppose u is convex. From Jensen's inequality, u is convex iff

$$\sum_n \lambda_n \big(-u(x_n)\big) \leq -u\left(\sum_n \lambda_n x_n\right)$$

iff

$$\sum_n \lambda_n \big(u(x_n)\big) \geq u\left(\sum_n \lambda_n x_n\right),$$

iff the DM is risk-loving.

Concavity and Risk Aversion

Proof of Theorem B: From Theorem A, if the DM is risk-averse, then $E_p u(x) \leq u(E_p x)$. By definition, $u(CE_p) = E_p u(x) \leq u(E_p x)$, and since u is increasing, $CE(p) \leq E_p x$.

Back

Comparing Attitudes to Risk

1 iff 2: $u_2 = g \circ u_1$ if and only if for all p,
$E_p u_2 = E_p g \circ u_1 \leq g(Eu_1)$. Then

$$\begin{aligned} u_2\big(CE_2(p)\big) &= E_p u_2 = E_p g \circ u_1 \\ &\leq g\big(E_p u_1\big) = g \circ u_1\big(CE_1(p)\big) = u_2\big(CE_1(p)\big). \end{aligned}$$

Since u_2 is increasing, $CE_2(p) \leq CE_1(p)$.

1 iff 3: Since u_1 and u_2 are increasing functions, there is an increasing function g such that $u_2 = g \circ u_1$. The chain rule implies that $r_2 = r_1 - g''/g'$, so $g'' = (r_1 - r_2)g'$. Since $g' > 0$, $g'' \leq 0$ iff $r_2 \geq r_1$.

Back