Ordinal Representations

September 19, 2002

Note: The marked exercises are not (yet) homework assignments. They are additional things I thought it would amuse you to think about.

1 What is an ***Ordinal Representation***?

We are given a *preference order* $\succ$ on X.

Definition 1. *A* utility representation *of the preference order* $\succ$ *is a function* $U : X \to \mathbf{R}$ *such that* $x \succ y$ *if and only if* $u(x) > u(y)$.

What do we mean by an ordinal representation? First, a representation is a numerical scaling — a thermometer to measure preference. Thus if x is better than y, x gets a higher utility number than y, just as if New York City is hotter than Boston, NY gets a higher temperature number. But with utility, only the ordinal ranking matters. Temperature is not an ordinal scale. New York is only slightly hotter than Boston, while Miami is much hotter.

$$T(\text{Boston}) - T(\text{Miami}) > T(\text{Boston}) - T(\text{New York}) > 0$$

The temperature difference between New York and Boston is smaller than the temperature difference between Miami and Boston. But to say that

$$u(x) - u(y) > u(a) - u(b) > 0$$

does not mean that x is more preferred to y than a is to b. We express this as follows:

Definition 2. *A utility representation for* $\succ$ *is* ordinal *iff for any strictly increasing function* $f : \mathbf{R} \to \mathbf{R}$, $f \circ U$ *is also a utility representation for* $\succ$.

Proposition 1. *A utility representation for a preference order* $\succ$ *is ordinal.*

Proof. If f is strictly increasing and U is a utility representation for $\succ$, then $x \succ y$ iff $U(x) > U(y)$ iff $f\big(U(x)\big) > f\big(U(y)\big)$. □

2 Why do we want an ordinal representation?

Summary: An ordering is just a list of pairs, which is hard to grasp. A utility function is a convenient way of summarizing properties of the order. For instance, with expected utility preferences of the form $U(p) = \sum_a u(a)p_a$, risk aversion — not preferring a gamble to its expected value — is equivalent to the concavity of u. The curvature of u measures how risk-averse the decision-maker is.

Optimization: We want to find optimal elements of orders on feasible sets. Sometimes these are more easily computed with utility functions. For instance, if U is C^1 and B is of the form $\{x : F(x) \leq 0\}$, then optima can be found with the calculus.

So why not start with utilities?

- We don't know that utility representations exist.
- Some characteristic properties of classes of preferences are better understood by expressing them in terms of orderings.

3 When do ordinal representations exist?

There are really two questions to ask:

- Does every preference order have a representation? More generally, what binary relations have numerical representations?
- Does every function from X to $\mathbf{R}$ represent some preference order? That is for a given $U : X \to \mathbf{R}$, define $x \succ_U y$ iff $U(x) > U(y)$.

The answer to these questions depends on the cardinality of X and the properties of $\succ$. Recall that an asymmetric relation $\succ$ is a

partial order: if it is *transitive*;

preference order: if it is negatively transitive;

linear order: if it is a partial order for which $\sim$ is the identity relation.

3.1 Denumerable X

3.1.1 Preference orders

For denumerable sets, every preference order has a representation. Recall K. Proposition 2.3; in particular, if $\succ$ is a preference relation, it is transitive and irreflexive. Also recall K. Proposition 2.4d: If $w \succ x$, $x \sim y$, and $y \succ z$, then $w \succ y$ and $x \succ z$.

Theorem 1. *Suppose X is denumerable.*

1. *If $\succ$ is a preference order, then it has a utility representation.*
2. *If $U : X \to \mathbf{R}$, then $\succ_U$ is a preference order.*

Proof.
2. Asymmetry is obvious. To check negative transitivity, suppose that not $x \succ_U y$ and not $y \succ z$. Then $U(x) \geq U(y)$ and $U(y) \geq U(z)$, so $U(x) \geq U(z)$, so not $z \succ_U y$. This argument holds in any domain, so we will not revisit it.

1. This is the interesting claim. We will make use of K. Proposition 2.4.d — in particular, if $x \sim y$ and $y \succ z$, then $x \succ z$. The art of the proof is to define a candidate utility function and see that it works.

Begin by indexing X: $X = \{x_1, x_2, \ldots\}$, and consider a preference order $\succ$. For each $x \in X$ define $W(x) = \{y : x \succ y\}$, the "worse than x" set. Define $N(x) = \{n : x_n \in W(x)\}$; the set of indices of elements in the worse than x set. Finally, define

$$U(x) = 0 + \sum_{n \in N(x)} \left(\frac{1}{2}\right)^n$$

We must show that U is a utility representation for $\succ$; that is, $U(x) > U(y)$ if and only if $x \succ y$.

Suppose that $x \succ y$. Since $\succ$ is transitive and irreflexive, $W(y) \subsetneq W(x)$. Consequently $N(y) \subsetneq N(x)$, and so $U(x) > U(y)$.

Suppose that $U(x) > U(y)$. There are only three possibilities for the order of x and y: $x \succ y$, $x \sim y$ and $y \succ x$. We will rule out the last two. The third is ruled out, because we have already shown that $y \succ x$ implies $U(y) > U(x)$. Suppose $x \sim y$. If $z \in W(y)$, then 2.4.d implies that $z \in W(x)$ and vice versa. Thus $N(x) = N(y)$ and so $U(x) = U(y)$. The only remaining possibility is $x \succ y$. □

3.1.2 Partial orders

Indifference need not be transitive in a partial order, so there is no possibility of getting a full numerical representation. In the following figure, most preferred items are at the top, and there is a link down from one node to another if the higher node is preferred to the lower. The ordering is then completed by transitivity. In this figure, $a \sim b$, $b \sim c$ and $a \succ c$. If $\succ$ had an ordinal

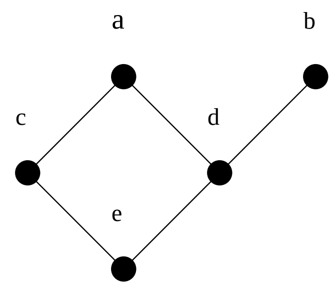

Figure 1: A Partial Order.

representation U, then it would follow that $U(a) = U(b)$, $U(b) = U(c)$, and $U(a) > U(c)$, which is impossible. However, it has a representation in the following weaker sense:

Definition 3. *A* (weak) utility representation *of the partial order* $\succ$ *is a function* $U : X \to \mathbf{R}$ *such that if* $x \succ y$*, then* $U(x) > U(y)$*.*

Theorem 2. *Suppose* X *is denumerable. If* $\succ$ *is a partial order, then it has a weak utility representation.*

Proof. The same construction as that in the proof of Theorem 1 works here. □

Notice that if $\succ$ is a partial order, then $C(B, \succ)$ exists for all $B \in P(X)$, and $C(B, \succ)$ is the set of all elements that maximize U on B.

Exercise 1. *Which of Sen's axioms* α *and* β *fail to hold? Find axioms which characterize those* $C(B)$ *which are a* $C(B, \succ)$ *for some partial order* $\succ$*.*

Exercise 2. *Let* $\succ$ *be a partial order on a denumerable set* X*. Define* $\succeq$ *and* $\sim$ *in the usual way. Define* $x \approx y$ *if for all* z *such that* $x \approx z$*,* $y \approx z$*. Show*

1. $\approx$ *is an equivalence relation.*
2. *If* $w \approx x$*,* $x \succ y$*, and* $y \approx z$*, then* $w \succ y$ *and* $x \succ z$*.*
3. *There is a function* $U : X \to \mathbf{R}$ *such that if* $x \succ y$*, then* $U(x) > U(y)$ *and* $x \approx y$ *iff* $U(x) = U(y)$*.*

Does this still hold true if $\succ$ *is only acyclic rather than transitive?*

3.2 Uncountable X

Not all preference orders are representable.

Example:
Let $X = \mathbf{R}^2_+$. Define the relation $(x_1, x_2) \succ (y_1, y_2)$ iff $x_1 > y_1$ or $x_1 = y_1$ and $x_2 > y_2$. It is called the *lexicographic order* on $\mathbf{R}^2$. In Figure 1, better

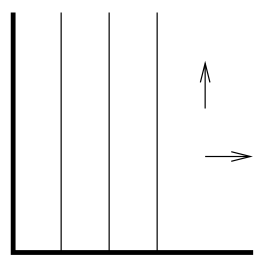

Figure 2: The Lexicographic Order.

points are to the right, but if two points are equally far to the right, the top point is better. This order has no utility representation. To see why, choose two distinct points on each vertical line. Suppose there were a utility representation U. The top point t_x on the line with first coordinate x must map to a higher number than the bottom point b_x on that line. Now consider the collection of intervals $\{[U(b_x), U(t_x)] : x \geq 0\}$. These intervals are all disjoint. Furthermore, since they are non-degenerate, each contains a rational number. These rational numbers are all distinct, and we have one for each vertical line, so if a utility function exists, there must exist an uncountable collection of rational numbers. No such collection exists; the rationals are countable. So U must not in fact exist. □

Exercise 3. *Show that the lexicographic order is in fact a preference order.*

3.2.1 Existence of ordinal representations

Another example will illustrate what an ordering that has an ordinal representation looks like.

Example:
Take X to be $\mathbf{R}^{\mathbf{2}}_{+}$. For each $x \in X$, define $l(x)$ to be the line with slope -1 through x intersected with X. Define $x \succ y$ if y lies above the line $l(x)$. The situation is illustrated in figure 3. Point y is preferred to point x because

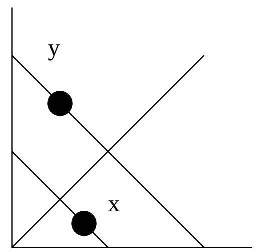

Figure 3: A representable order.

y lies above $l(x)$. It is easy to see that $\succ$ is a preference order. It is also easy to see that $y \sim x$ if and only if $y \in l(x)$. The lines with slope -1 are called *indifference curves*, since two points on the same line are indifferent to each other. Ordering the points comes down to ordering the indifference curves. Lines farther out are better, so a natural utility representation is to measure how far each line is from the origin; that is, where it intersects the diagonal. □

For a utility representation to exist, the order $\succ$ on X must "look like" the $>$ order on the real line. The order $\geq$ is complete, transitive and reflexive, and so is $\succeq$ for any preference order $\succ$. The $\geq$ order on $\mathbf{R}$ has another property that, strictly speaking, has to do with the structure of $\mathbf{R}$ as well as the order. The rational numbers $\mathbf{Q}$ are a countable subset of $\mathbf{R}$ with the property that if a, b are in $\mathbf{R}/\mathbf{Q}$ and $a > b$, then there is a rational number $r \in \mathbf{Q}$ such that $a > r > b$. It is exactly this property that fails in the lexicographic example.

Definition 4. *A set $Z \subset X$ is* order-dense *if and only if for each pair of elements $x, y \in X/Z$ such that $x \succ y$ there is a $z \in Z$ such that $x \succ z \succ y$.*

Theorem 3. *For a preference order $\succ$ on X, a utility representation exists if and only if X contains a countable order-dense subset.*

Proof sketch: Essentially the denumerable construction works: let Z denote the countable order-dense set, and let $N(x)$ denote the set of indices of elements of Z that are worse than x. Proceed as before. □

The existence of a countable order-dense set is an example of an *Archimedean assumption*. It is required so that the preference order "fits in" to $\mathbf{R}$. The set $\mathbf{R}$ is an example of an *ordered field*. The rational numbers are another example. There are also ordered fields that strictly contain $\mathbf{R}$ — the so-called hyperreal or non-standard numbers. One can show that if $\succ$ is any preference relation, it can be represented in some ordered field. If X is uncountable, it certainly cannot be represented in $\mathbf{Q}$, and in order to fit into $\mathbf{R}$, it must be "small enough". This is what order-denseness does.

Exercise 4. *State and prove a representation theorem for partial orders on a non-denumerable X.*

Clearly lexicographic preferences have no countable order-dense set, since any order-dense set must contain at least one element on each vertical line, and there are an uncountable number of such lines. The points in $\mathbf{R}^{\mathbf{2}}_+$ with rational coordinates are order-dense for $\succ$ in the second example.

3.2.2 Continuous representations

The point of choice theory is to describe choice behavior by deriving the choice functions $C(B, \succ)$. When X is finite, or each B we care about is finite, the fact that $\succ$ is a preference order is enough to derive that $C(B, \succ) \neq \emptyset$. When B is not finite, choice functions may be empty.

Example:
X is the set of non-negative integers. $x \succ y$ iff $x > y$. B is the set of even integers. □

So we want to find restrictions on $\succ$ and on the set of admissible $\mathcal{B}$ of admissible feasible sets B such that $C(B, \succ) \neq \emptyset$ for all all $B \in \mathcal{B}$. For example, if X is denumerable and $\mathcal{B}$ is taken to be the collection of all non-empty finite subsets of X, K. Proposition 2.8 still holds: If $\succ$ is a preference, then $C(B,\succ) \neq \emptyset$.

When X is not denumerable, more assumptions are needed. The setting that comes up most often in modelling applications has X a closed subset of a Euclidean space. If $\succ$ has a utility representation, then

$$C(B,\succ) = \operatorname{argmax}\{U(x), x \in B\}$$

We would like to know conditions on U and B that will guarantee the existence of solutions to this problem.

A natural generalization of finiteness to this setting is *compactness.*

Definition 5. *A set B in* $\mathbf{R^n}$ *is* compact *iff it is both closed and bounded.*

A basic fact of real analysis is **Weierstrass' Theorem:** Every continuous function has a maximum on every compact set. Formally, if U is continuous and B is compact, then there is an $x \in B$ such that for all $y \in B$, $U(x) \geq U(y)$. So if we're willing to accept the restriction that $\mathcal{B}$ contains only compact sets, then a sufficient condition guaranteeing choice is that $\succ$ have a continuous utility representation. What conditions on $\succ$ guarantee that it has a continuous utility representation?

Recall that a preference order is just a set of pairs of alternatives: $\{(x,y) \in X \times X : x \succ y\}$.

Definition 6. *A preference order $\succ$ is* continuous *iff $\{(x,y) \in X \times X : x \succ y\}$ is open in $X \times X$.*

Theorem 4. *A preference order has a continuous utility representation iff it is continuous.*

Proof. See Debreu (1954). A cleaner discussion can be found in Rader (1963). □

Exercise 5. *Show that if $\succ$ is open, the sets $W(x)$ and the corresponding "better than" sets $B(x) = \{y : y \succ x\}$ are open for all $x \in X$. Is the converse true?*

4 Characterizing preferences through their representations

Another aspect of representation theory is the characterization of preferences with certain kinds of representations.

Example:
For instance, consider choice under uncertainty. Suppose there are a finite set of rewards $R = \{r_1, \dots r_n\}$. A *lottery* is a probability distribution on rewards; that is, a vector $(p_1, \dots, p_n)$. Decision makers have preferences on lotteries. A utility function U on lotteries is an *expected utility* representation if there is a function $u : R \to \mathbf{R}$ such that

$$U(p_1, \dots, p_n) = p_1 u(r_1) + \cdots + p_n u(r_n)$$

We would like to characterize or otherwise identify those preference orders that have an expected utility representation. □

This is just one example of how we might like to identify a class of preferences based on properties of a numerical representation. Another example, which sits apart from choice under uncertainty, follows.

4.1 Additive Separability

The theory presented so far treats objects of choice as primitive abstract entities. But in real choice problems the objects of choice have structure, and this structure may suggest meaningful restrictions on preferences. Here I want to think of objects of choice as bundles of attributes. The classic example of this is the *commodity bundle* in economic analysis. When I go to the grocery store I don't just choose coffee or tea. I also have to choose lemon or sugar, milk or cream, etc. If the store has no fresh lemons, I may choose to put coffee rather than tea into my shopping basket. At a good restaurant one puts together an entire meal from a list of appetizers, first courses, entrees and desserts. One chooses the meal, but each possible meal is described by a list of these attributes. Choice under uncertainty offers another example of this phenomenon, which will be discussed at the end of this section.

How much utility do I get from a box of Kellogg's Corn Flakes? It is hard to answer this question because how much I like my corn flakes depends upon whether we have milk in the fridge, and what bugs are living in the sugar bowl. I never consume cereal alone, but only as part of a breakfast meal. I have to consider all of the attributes together, and for breakfast I cannot value one attribute independently of the others. Nonetheless one can imagine situations where it may be sensible to value each object independently. Suppose you are buying health insurance. You can describe the policy by listing all of the possible health events that could happen to you, and the net payout from the policy in each event. Thus a policy is just a list of attributes. Here it is plausible that you could talk meaningfully of the value of the surgical coverage, or the value of the prescription drug coverage. That is, one can talk meaningfully about preferences over each attribute, and think about aggregating them to get aggregate preferences over policies.

In formalizing this idea, objects of choice may be thought of as bundles of attributes. Cars may be characterized by gas milage, engine power, quality of the ride, etc. Utility of a given car depends upon the whole bundle of characteristics, but if the characteristics are independent, we may be able to sensibly ask after the value of gas milage, and so forth. When we can, utility is said to be *additive* in the attributes. The general question is, when objects of choice can be described by a collection of factors, when can one define utility on each factor, and when is utility of choice objects additive in the utilities of the factors. Expected utility is a particular example of this, but far from the only example.

Suppose that X is a product space: $X = X_1 \times \cdots \times X_n$. Each $x \in X$ is a bundle of attributes or characteristics. Each X_i is a *factor*. Suppose for concreteness that each X_i is an interval in $\mathbf{R}$. Given is a complete weak order $\succeq$ on X.

Definition 7. *A utility function on X which represents $\succeq$ is* additively separable *if there are functions $u_i : X_i \to \mathbf{R}$ such that*

$$u(x) = u_1(x_1) + \cdots + u_n(x_n)$$

Why does additive separability make sense?

Additive separable representations are "more nearly unique" than ordinal representations. If $U : X \to \mathbf{R}$ is an additive separable representation of $\succ$ and $f : \mathbf{R} \to \mathbf{R}$ is strictly increasing, then $f \circ U$ is a utility representation of $\succ$, but it is not necessarily additively separable.

Theorem 5. *Suppose $U : X \to \mathbf{R}$ is an additively separable representation of $\succ$. The function $V : X \to \mathbf{R}$ is an additively separable representation for $\succ$ iff there are real numbers $a > 0$ and b such that $V = aU + b$.*

This theorem is not true for arbitrary X and U. It requires that the image of X under U be rich enough. See Basu (1982) on this point. The conditions on X we suppose and the conclusions about U we derive will be sufficient to reach this conclusion.

Suppose we can write $X = \prod_{i=1}^{n} X_i$, where each X_i is a connected subset of some Euclidean space. Suppose that $\succ$ is a preference order for which, for all $x \in X$, both $W(x)$ and $B(x)$ are open.

For any subset I of indices and any element $x \in X$, write $x_I = (x_i)_{i \in I}$. Write x_{-i} when referring to the set of all indices but i. Define the preference order $\succ_{x_{I^c}}$ on $\prod_{i \in I} X_i$ such that $a \succ_{x_{I^c}} b$ iff $(a, x_{I^c}) \succ (b, x_{I^c})$. Think of these orders as preferences conditional on receiving the factors x_{I^c}.

Definition 8. *The factors of X are* independent *if for all I and $x, y \in X$, $\succ_{x_I} = \succ_{y_I}$. Factor i is* essential *if there is an x_{-i} such that $\succ_{x_{-i}}$ is non-empty.*

Theorem 6. *Suppose $\succ$ is a preference order such that the n factors are independent and there are at least three essential factors, then $\succ$ has an additive representation. Each u_i is continuous. The representation is unique up to positive affine transformations.*

Proof. See Debreu (1960) □

This approach to additive separability hides the algebraic structure of the problem in topological assumptions. What guarantees, for instance, the existence of an additive separable representation on a finite set of alternatives?

Here is the "standard" approach, laid out for two factors. Suppose $X = X_1 \times X_2$, and that $\succ$ is a binary relation on X which satisfies the following conditions:

A.1. (preference order): $\succ$ is asymmetric and negatively transitive.

A.2. (independence): For all a, b in X_1 and p, q in X_2, if $ap \succ bp$ then $aq \succ bq$, and if $ap \succ aq$ then $bp \succ bq$.

A.3. (Thomsen): For all $a, b, c \in X_1$ and $p, q, r \in X_2$, if $bp \sim aq$ and $cp \sim ar$, then $cq \sim br$.

A.4. (essential): Both factors are essential.

A.5. (solvability): For $a, b, c \in X_1$ and $p, q, r \in X_2$, if $ap \succeq bq \succeq cp$, then there is an $x \in X_1$ such that $xp \sim bp$, and if $ap \succeq bq \succeq ar$, there is a $y \in X_2$ such that $ay \sim bq$.

A.6. (Archimedes): An Archimedean axiom.

Definition 9. *A pair $(X, \succ)$ is an* additive preference structure *if $X = X_1 \times X_2$ and $\succ$ satisfies axioms A.1–6.*

Theorem 7. *If X is an additive preference structure, then $\succ$ has an additively separable representation, and that representation is unique (among additively separable representations) up to positive affine transformations. If $\succ$ on $X = X_1 \times X_2$ has an additively separable representation, then $\succ$ satisfies A.1–3.*

Proof. A clean proof can be found in Holman (1971). □

The Thomsen condition captures the essence of additive separability. It is easy to check its' necessity. It describes a kind of "parallel property" that indifference curves must have. The condition can be described in the figure below.

This figure contains three pairs of points, identifiable by their shading. The Thomsen condition says that if the two points are indifferent in any two of the pairs, the two points in the third pair are indifferent as well. If an indifference curve runs through the two black points, and another runs through both grey points, then a third curve runs through through the two white points.

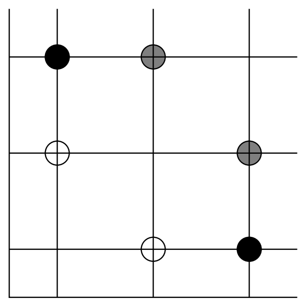

Figure 4: The Thomsen condition.

The Thomsen condition is a statement about how different indifference curves fit together. To see the implications of additive separability for how indifference curves should fit together somewhat differently, take X to be the non-negative orthant of the Euclidean plane, and suppose $\succ$ has a utility representation $U(x, y) = f(x) + g(y)$, and all functions are C^1. The indifference curve corresponding to utility level u is the set of solutions to the equation

$$f(x) + g(y) = u$$

Differentiating implicity, the derivative of the indifference curve in the xy-plane through the point (x, y) is $y'(x) = -f'(x)/g'(y)$. Consider the points A and B in the figure below. The ratio of the slope of the curve through A to that of the curve through B is $g'(y_1)/g'(y_2)$. This expression is independent of x. The points C and D have the same y coordinates as A and B, respectively. So the ratio of the slope of the curve through C to that of the curve through D should be identical. A similar condition must hold for points A and C, and B and D.

Exercise 6. *Take $X = \mathbf{R}^2_+$, and define $U(x, y) = x^2 + xy + y^2$. The function U represents some preference order, and U is not additively separable. Does* *the preference order U represents have an additively separable representation? Answer the same question for $V(x,y) = x^2 + 2xy + y^2$. Finally, consider $U_\alpha(x,y) = x^2 + \alpha xy + y^2$ for $\alpha \geq 0$. For which values of the parameter α does the preference order represented by U_α have an additively separable representation?*

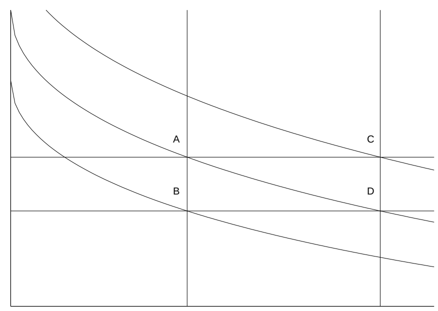

Figure 5: Additive separability slope conditions.

I began this section by claiming that additive separability sits apart from choice under uncertainty. Strictly speaking, this is false. Although additive separability is interesting in many situations where uncertainty plays no role, it has connections to choice under uncertainty as well. A simple example follows. Let's consider bets on whether or not W. will be reelected. A bet can be described by a pair of numbers: what you win if he is reelected, and what you will win if he is not. So for instance, the bet $(10, -10)$ pays off \$10 if W. wins and $-\$10$ if he does not (that is, you pay \$10 if he loses). The bet $(0, 0)$ is "no bet". The set of all possible bets is $\mathbf{R^2}$, and a typical bet is the pair (z_1, z_2). An expected utility representation for a preference order $\succ$ on the set of all bets is a pair (p, u) where p is a probability of W. winning, and $u : \mathbf{R} \to \mathbf{R}$ is a real valued function, such that

$$(x_1, x_2) \succ (y_1, y_2) \qquad \text{iff} \qquad pu(z_1) + (1-p)u(z_2) > pu(y_1) + (1-p)u(y_2)$$

That is, p and u are such that the function $U(z_1, z_2) = pu(z_1) + (1-p)u(z_2)$ is a utility representation for $\succ$. Notice that the utility function U is additively separable in its components z_1, z_2. In this case, expected utility is a special case of additive separability on an appropriate set X of choices.

Exercise 7. *Consider the utility function* $U(z_1, z_2) = \min(z_1, z_2)$*, which in the uncertainty context gives rise to the* maximin criterion*. Which of the assumptions in Theorem 7 does its ordering violate?*

5 Preference Intensity and Cardinal Utility

5.1 Cardinal Utility

The theory of preference and choice behavior presented to this point is choice based. Its sole purpose is to generate choice functions, and the mode of inquiry for examining preferences is through revealed choice data. The utility theory for any choice-based theory of preferences must be ordinal, because any transformation of a utility function which preserves rankings will exhibit the same choice behavior. A stronger invariance principle than ordinality is satisfied by measures such as temperature and length.

Definition 10. *A representation of a preference order* $\succ$ *on an alternative set* X *is* scale invariant *or* cardinal *if, whenever both* $u : X \to \mathbf{R}$ *and* $v : X \to \mathbf{R}$ *represent* $\succ$*, they are related by positive affine transformations: That is,* $v = au + b$ *where* a *and* b *are real numbers and* a *is strictly positive.*

Particular *classes* of representations are scale invariant. For instance, on a rich enough X, the set of all additively separable representations of a given preference order is closed only under positive affine transformations whenever it exists. But this is not the same as saying that utility is cardinal, for any strictly increasing transformation of an additively separable representation which is not positive affine will not be additively separable, but it will represent $\succ$. Cardinal theories can only arise by moving beyond choice-based theories.

One circumstance in which cardinal utility can be derived is that when subjects are allowed to express an intensity of preference. It is natural to say "I prefer carrots to peas and steak over beef liver, but I prefer steak over liver more than I prefer carrots over peas". "I have a strong preference for John Thomas' Steak House over the Moosewood, but only a mild preference for

Dano's over Willow." Here I will show how elicitations of preference intensity lead to a cardinal, or scale invariant, utility theory.

As before, X denote the set of alternatives. The cartesian product $X \times X$ is the set of all pairs of alternatives. The primitive order of this theory of preference intensity is a binary relation $\rhd$ on $X \times X$. The expression $a \rightsquigarrow b \rhd c \rightsquigarrow d$ is interpreted as meaning "a is preferred over b more than c is preferred over d". The weak order for $\rhd$ is denoted $\unrhd$, and the indifference relation is $\equiv$. In ranking the points (a, b) and (c, d) of X, with $\rhd$, I will write $a \rightsquigarrow b \rhd c \rightsquigarrow d$ to emphasize that the difference of b from a is being compared to the difference of d from c. (Note: The minus sign $-$ makes us think of subtraction, and the minus in a circle $\ominus$ which I used in class makes some of us think of a binary operation. The symbol $\rightsquigarrow$ seems like a serviceable compromise.)

The relationship $\rhd$ satisfies the following conditions:

A.1. (preference order): $\rhd$ is asymmetric and negatively transitive.

A.2. (reverse consistency): If $a \rightsquigarrow b \rhd c \rightsquigarrow d$, then $d \rightsquigarrow c \rhd b \rightsquigarrow a$.

A.3. (concatenation): If $a \rightsquigarrow b \rhd a' \rightsquigarrow b'$ and $b \rightsquigarrow c \rhd b' \rightsquigarrow c'$, then $a \rightsquigarrow c \rhd a' \rightsquigarrow c'$.

A.4. (Archimedes): An Archimedean axiom.

A.5. (solvability): If $a \rightsquigarrow b \unrhd c \rightsquigarrow d \unrhd a \rightsquigarrow a$, then there are alternatives x, y in X such that $a \rightsquigarrow x \equiv c \rightsquigarrow d$ and $y \rightsquigarrow b \equiv c \rightsquigarrow d$.

Definition 11. *A pair* $(X, \rhd)$ *is a* cardinal preference structure *if* $\rhd$ *is a binary relation on* $X \times X$ *satisfying axioms A.1–4.*

Axioms A.1–3 are intuitive. Axiom A.1 is equivalent to requiring that the $\unrhd$ relation is a weak order, that is, reflexive, transitive and complete. Thought of this way, a moment's reflection suggests the reasonability of this requirement.

Axiom A.2 says that if I prefer a over b more than I prefer c over d, then if I measure preference intensity in the other direction, my intensities should be reversed.

Axiom A.3. says that if the gap from a down to b is large, and the gap from b down to c is large, then the gap from a down to c should be large. I find this intuitive but others may disagree. Samuelson (1938), in particular, considers this assumption to be "infinitely improbable".

Axiom A.4., solvability, requires a rich enough alternative set and lots of strict comparisons. If we were making topological assumptions, solvability would follow from requirements that $\triangleright$ be *open*. It is not terribly intuitive.

Archimedean assumptions such as the yet-to-be-stated A.5. are never intuitive, but their role is clear.

The order $\triangleright$ compares preference intensity changes in moving from one choice to another, and it would be natural to measure these intensity changes in terms of utility differences.

Definition 12. *The cardinal preference* $(X, \triangleright)$ *has a* utility difference representation *if there is a function* $U : X \to \mathbf{R}$ *such that* $a \rightsquigarrow b \triangleright c \rightsquigarrow d$ *iff* $U(a) - U(b) > U(c) - U(d)$.

The representation theorem is:

Theorem 8. *If* $\triangleright$ *is a solvable relation on* $X \times X$ *the following are equivalent:*

1. $\triangleright$ *satisfies A.1–A.4 (that is,* $(X, \triangleright)$ *is a cardinal preference structure).*
2. $(X, \triangleright)$ *has a cardinal utility difference representation.*

Proof. This result is Theorem 4.2 of Krantz, Luce, Suppes & Tversky (1971). Their proof is not helpful, and a good challenge for graduate students is to provide a direct proof. □

A few implications of the axioms make it clear why such a theorem should be true, and also help explain some of the axioms.

First, all "0 differences" are equivalent.

Proposition 2. *For all* a *and* b *in* X, $a \rightsquigarrow a \equiv b \rightsquigarrow b$.

Proof. This is an immediate consequence of reverse consistency. Suppose, without loss of generality, that $a \rightsquigarrow a \trianglerighteq b \rightsquigarrow b$. Applying reverse consistency to this statement gives $b \rightsquigarrow b \trianglerighteq a \rightsquigarrow a$. $\square$

This helps interpret the solvability axiom. The claim that the equation in x, $a \rightsquigarrow x \equiv c \rightsquigarrow d$, has a solution, because $a \rightsquigarrow b \trianglerighteq c \rightsquigarrow d \trianglerighteq a \rightsquigarrow a$. The claim involving the y equation seems less intuitive, but since $a \rightsquigarrow a \equiv b \rightsquigarrow b$, an equivalent statement is that $a \rightsquigarrow b \trianglerighteq c \rightsquigarrow d \trianglerighteq b \rightsquigarrow b$ implies that the equation $y \rightsquigarrow b \equiv c \rightsquigarrow d$ has a solution.

Next, recall Definition 8 of *independent factors*, that all possible values of each component of the elements of $X \times X$ can be ordered by $\triangleright$ in a way which is independent of the value of the other component.

Proposition 3. *The factors are independent.*

Proof. This is a consequence of the preference order and concatenation axioms. Suppose $a \rightsquigarrow x \trianglerighteq b \rightsquigarrow x$, and choose $y \in X$. Since $x \rightsquigarrow y \trianglerighteq x \rightsquigarrow y$, $a \rightsquigarrow y \trianglerighteq a \rightsquigarrow y$. And conversely. $\square$

Define the relations $\succ_1$ and $\succ_2$ on X as follows: $a \succ_1 b$ if there is an x such that $a \rightsquigarrow x \triangleright b \rightsquigarrow x$; and $a \succ_2 b$ if there is a y such that $y \rightsquigarrow a \triangleright y \rightsquigarrow b$. Factor independence and the preference order axiom imply that these relations are preference orders on X. And in fact $\triangleright_1$ is an intuitive notion of (ordinal) preference. Fix a reference bundle x. I prefer a to b if and only if I prefer a over the reference bundle more than I prefer b over the reference bundle. Independence says that this notion of preference is independent of the reference bundle. Perhaps a more intuitive notion of ordinal preference would be to say that a is preferred to b if and only if $a \rightsquigarrow b \triangleright a \rightsquigarrow a$. Proving that this relationship satisfies the preference order action is a nice exercise.

A final proposition suggests why a difference representation is plausible. It states that $\succ_2$ is the reverse of $\succ_1$.

Proposition 4. $a \succ_1 b$ *if and only if* $b \succ_2 a$.

Proof. This is a consequence of reverse consistency. If $a \succ_1 b$, then for any $x \in X$, $a \rightsquigarrow x \rhd b \rightsquigarrow x$. Reverse consistency implies that $x \rightsquigarrow b \rhd x \rightsquigarrow a$, and so $b \succ_2 a$. The proof of the converse is identical. $\square$

Why is cardinal utility plausible from these axioms? First, suppose that $U : X \to \mathbf{R}$ is a utility representation of $\succ_1$. Proposition 4 implies that $-U$ is a utility representation of $\succ_2$. Next, suppose that $a \sim_1 b$ and $c \sim_2 d$. Then from the definitions, Proposition 3 and the preference order axiom, $a \rightsquigarrow c \equiv b \rightsquigarrow c \equiv b \rightsquigarrow d$. Putting this together, if $U(a) = U(b)$ and $U(c) = U(d)$, then $a \rightsquigarrow c \sim b \rightsquigarrow d$. Our axioms imply that $\rhd$ has a utility representation $V : X \times X \to \mathbf{R}$, and we have just seen that $V(a \rightsquigarrow c) = V(b \rightsquigarrow d)$ whenever $U(a) = U(b)$ and $U(c) = U(d)$. Thus V must be of the form

$$V(a \rightsquigarrow b) = F\big(U(a), U(b)\big)$$

for some $F : \mathbf{R^2} \to \mathbf{R}$. We also know from Proposition 2 that F must be constant on the diagonal. Finally, F must be strictly increasing in its first argument and strictly decreasing in its second. To see this, choose u, v, w in the range of U; that is, there are $x, y, z \in X$ such that $U(x) = u$, and so forth. Suppose that $u > v$. Then $x \rightsquigarrow z \rhd y \rightsquigarrow z$, so $F(u, w) > F(v, w)$. A similar argument works for the second component.

This is suggestive, but it takes some work to show that the right U can be chosen such that F then takes the form $F(u, v) = u - v$.

Theorem 8 and the ensuing discussion shows that every cardinal preference structure has embedded in it a preference order $\succ$ on X, and that $\rhd$ has a numerical representation of the form $V(a \rightsquigarrow b) = U(a) - U(b)$ where U represents $\succ$. It is a mantra of sorts that any transformation of U which preserves the ordering of utility differences must be positive affine. In other words, U is cardinal. Strictly speaking, this is false. For example, suppose $X = \{a, b, c\}$, with $a \rightsquigarrow b \rhd b \rightsquigarrow c \rhd a \rightsquigarrow a$, filling in the rest with the preference order, reverse consistency and concatenation axioms. Fix $U(b)$ and $U(c)$. If $U(a)$ any sufficiently large number, $V(x \rightsquigarrow y) = U(x) - U(y)$ represents $\rhd$. This idea generalizes.

Proposition 5. *For any $n < \infty$ there is a cardinal preference structure $(X, \rhd)$ with $|X| = n$ with utility difference representations U and V that are not positive affine transformations of each other.*

Proof. Take $X = \{1, \ldots, n\}$ and define $U(x) = 10^x$. Let $\rhd$ be the relation defined by the equivalence $a \rightsquigarrow b \rhd c \rightsquigarrow d$ iff $U(a) - U(b) > U(c) - U(d)$. Each indifference class of $\rhd$ contains exactly one pair. To see this observe that $a \rightsquigarrow b \equiv c \rightsquigarrow d$ iff $10^a - 10^b = 10^c - 10^d$ iff $10^a + 10^d = 10^b + 10^c$. The left hand side describes the decimal representation of a number with a 1 in places a and d, and 0's elsewhere. And decimal representations are unique. Thus if $a \rightsquigarrow b \neq c \rightsquigarrow d$, the either $U(a) - U(b) > U(c) - U(d)$, or $U(a) - U(b) < U(c) - U(d)$. Thus the numbers $U(x)$ are an n-tuple that solve a finite number of strict inequalities. Since the inequalities are strict, any nearby numbers will also solve the inequalities. □

These examples are not solvable, so they do not contradict Theorem 8. Solvability forces X to be infinite if $\rhd$ is non-empty. In general, solvability implies that if $a \rightsquigarrow b \rhd b \rightsquigarrow b$, then the interval from b to a can be broken up into an arbitrary number of little pieces such that utility differences are the same (indifferent).

For a short, amusing discussion of the cardinal/ordinal utility debates in the economics literature of the 1930's, see Basu (1982).

5.2 Interpersonal Utility Comparisons

All too often we say things like, "You only slightly prefer the movie to the hockey game, while I want to see the game much more than I want to see the movie, so lets go to the game." The purpose of this statement is to aggregate two individuals' preferences into a single preference order. This aggregation problem has a long and celebrated history, going back to the 18th century.

This statement in quotes involves a cardinal interpersonal utility comparison. We can imagine four different preference aggregation problems, depending upon whether preferences are ordinal or cardinal, and on whether interpersonal utility comparisons are allowed or not. The "ordinal, not" case is a famous problem. It is the subject of Arrow's famous "general possibility theorem", which says that there is no reasonable way of aggregating ordinal preferences without interpersonal utility comparisons. *Social Choice* will be taken up later, so here is only a brief example to illustrate the problem.

Example: The Condorcet paradox:
Suppose three individuals and three alternatives. Preferences are : $a \succ_1 b \succ_1 c$, $c \succ_2 a \succ_2 b$ and $b \succ_3 c \succ_3 a$. Suppose preferences are aggregated by majority rule. Two individuals prefer a to be, so the aggregate order has $a \succ b$. Two prefer b to c, so the aggregate order has $b \succ c$. Unfortunately, two also prefer c to a, so the aggregate order has $c \succ a$, and fails to be transitive. □

In fact, the four cases mentioned above can be divided into several more cases, because one can be much more specific about what "interpersonal utility comparison" might mean. To make this clear, suppose that a set X of social alternatives is given, and also a set $\mathcal{N}$ of individuals. Each individual has a preference order $\succ_i$ on X. Suppose we already have utility representations for each individual's preference order on X. Define $u : X \times \mathcal{N} \to \mathbf{R}$ such that $x \succ_i y$ if and only if $u(x,i) > u(y,i)$. Let $\mathcal{D}$ denote the set of all such functions. Thus if $u \in \mathcal{D}$, $u(\,\cdot\,,i) : X \to \mathbf{R}$ is a representation for $\succ_i$. Social welfare functions assign a social ranking to each $u \in D$. To formalize this idea, let P denote the set of all preference orders on X.

Definition 13. *A* social welfare function *(SWF) is a map $\phi : \mathcal{D} \to P$.*

Why is the domain of SWFs preference orders rather than utility functions on X?

The various cardinality and interpersonal comparison properties can be expressed by asking after the set of transformations on $\mathcal{D}$ that leave the SWF ϕ invariant.

Definition 14. *A* transformation *of $u \in \mathcal{D}$ is a vector of functions $t = (t_i)_{i \in \mathcal{N}}$ where each $t : \mathbf{R} \to \mathbf{R}$. For $u \in \mathcal{D}$, define tu such that*

$$tu(x,i) = t\big(u(x,i)\big)$$

The SWF ϕ is invariant under t *if for all $u \in \mathcal{D}$, $\phi(tu) = \phi(u)$. For a set T of transformations, the SWF ϕ is* invariant under T *if for all $t \in T$, ϕ is invariant under t.*

Here are some examples. The SWF ϕ is

Ordinal and non-comparable (ONC): T is the set of all strictly increasing transformations.

Cardinal and non-comparable (CNC): $t \in T$ iff there are $a_i > 0$ and b_i such that $t_i(u) = a_i u + b$.

Ordinal and comparable (OC): $t \in T$ iff $t = (f, f, \dots, f)$ where f is a strictly increasing function on $\mathbf{R}$.

Cardinal and unit-comparable (CUC): $t \in T$ iff there are numbers $a > 0$ and b_i such that $t_i(u) = au + b_i$. The term is due to Sen (1970).

Cardinal and fully comparable (CFC): $t \in T$ iff there are numbers $a > 0$ and b such that $t_i(u) = au + b$.

All these invariance properties are different versions of scale invariance (or less). Another way to measure welfare is to fix a reference state, and compare welfare to that as a norm. Think of measuring temperature. We could fix the freezing point of water as a norm, and take that to be the 0 of any temperature scale. Temperature differences matter, to. Putting this together, if τ and τ' are two temperature scales, then there is an $a > 0$ such that $\tau' = a\tau$. Such invariance is called *ratio-scale invariance*. Do the same thing with SWFs.

Cardinal and normed (CN): $t \in T$ iff there is an $a > 0$ such that $t_i(u) = au$.

This class of utilities arises in axiomatic bargaining theory. Those familiar with the Nash bargaining solution will recall that utilities are measured relative to a reference point.

Exercise 8. *Order these classes by inclusion.*

Exercise 9. *Rawls (1971) argues that a just social preference order is one which ranks social states according the the criterion that $x \succ y$ iff*

$$\min_{i \in \mathcal{N}} u(x, i) > \min_{i \in \mathcal{N}} u(y, i)$$

To which of the classes described above does the Rawlsian social welfare function belong?

The goal of social choice theory is to characterize the ϕ in each of these classes. We are also interested in other properties, such as anonymity, non-dictatorship, and *Pareto optimality*. The last is an important property which comes up in a variety of situations.

Definition 15. *A social preference $\succ$ in P* respects Pareto optimality *if whenever for all i not $y \succ_i x$ and for some j, $x \succ_j y$, $x \succ y$.*

That is, if no individual prefers y to x, and someone prefers x to y, then x is socially better than y.

In order to carry out the social choice theory program, one first needs a measurement theory in which these definitions can all be posed. That is, appropriate social preference structures must be constructed. Much of this can be found in Sen (1970).

6 Appendix: Qualitative Probability

Representation theory is not confined to the representation of preference orders. The question is relevant for the numerical scaling of any order. Here I discuss the representation of "qualitative probability orders". These expressions of likelihood rankings could be derived from preferences, but they need not be. So they serve as an example of a measurement phenomenon to which representation theory is relevant which is not entirely preference based. In addition, qualitative probability is interesting in its own right as part of an understanding of choice under uncertainty. Students may find this interesting, but we do not expect to cover this material in class, at least in the near future.

Suppose we are given a collection $\mathcal{E}$ of *events*. It is natural to make statements such as "event A is more likely than event B." So we can talk about an ordering of likelihood. Write $A \succeq B$ to mean that "A is at least as likely as B". We frequently report likelihoods using probability distributions. "The probability of A is 1/2, while the probability of B is only 1/4." So a natural question to ask is, when can $\succeq$ be represented by a probability distribution?

There is a natural connection between preferences and qualitative probability. Suppose we have a set of events generated by the flips of, say, three coins. Suppose we offer you bets on the flips:

f_1: Win \$10 if any one coin turns up heads.

f_2: Win \$10 if any three coins turns up heads.

$\cdots$ etc.

If you announce that $f_1 \succ f_2$ then it is reasonable to conclude that you believe the event "one coin turns up heads" to be more likely than the event "two coins turn up heads". Of course, the preference order must satisfy some conditions for this to be a reasonable conclusion. See K. Chapter 8.

Eliciting preferences over bets is not the only way to generate a qualitative probability. For instance, you could compare two events A and B by flipping the coins ten times, and conclude that $A \succeq B$ if and only if A occurs at least as often as does B.

The basic set-up for qualitative probability is the following: We are given a set X of *states* and a collection $\mathcal{E}$ of subsets of X. A set $A \in \mathcal{E}$ is called an *event*, and it is events we order. The collection $\mathcal{E}$ is an *algebra* of sets. This means that $\mathcal{E}$ contains X and $\emptyset$, and is closed under the operations of union and complementation. (From this derive that $\mathcal{E}$ is also closed under intersection.)

Definition 16. *A* qualitative probability structure *is a triple* $(X, \mathcal{E}, \succ)$ *such that*

1. $\succ$ *is a preference order;*
2. $X \succ \emptyset$ *and for all* $A \in \mathcal{E}$, $A \succeq \emptyset$*;*
3. *if* A *is disjoint from both* B *and* C*, then* $B \succ C$ *iff* $A \cup B \succ A \cup C$*.*

Suppose for the nonce that $\mathcal{E}$ is finite. Then $\succ$ has an ordinal representation. There is a set function $p : \mathcal{E} \to \mathbf{R}$ such that $A \succ B$ iff $p(A) > p(B)$. In fact, however, we would like to be able to represent the likelihood order by a probability distribution. In particular, the representation should have the following *additive property*: If A and B are disjoint, then $p(A \cup B) = p(A) + p(B)$. That is, the representation maps disjoint union on the domain $\mathcal{E}$ into $+$ on the range. The other conditions one would want are that p is indeed a probability, that is, $p(\emptyset) = 0$ and $p(X) = 1$. It is clear that any positive affine transformation of p will also represent $\succ$, so any additive p can be rescaled to a probability measure.

Definition 17. *The qualitative probability structure* $(X, \mathcal{E}, \succ)$ *has a* probability representation *If there is a probability measure* p *on* $\mathcal{E}$ *such that* $A \succ B$ *iff* $p(A) > p(B)$.

One might think that additivity is a natural consequence of item 3 in the definition of a qualitative probability structure. That it was not was first shown in an example by Kraft, Pratt & Seidenberg (1959). A version of the example is discussed in K. Chapter 8. More illuminating than the example per se is a constructive method for generating it.

Example:
Let $X = \{a, b, c, d, e\}$. Suppose that the following claims are true:

$$\{a\} \succ \{b, c\} \qquad \{c, d\} \succ \{a, b\} \qquad \{b, e\} \succ \{a, c\} \tag{1}$$

Suppose a probability representation $p : X \to \mathbf{R}$ existed. Then p would have to satisfy the following linear inequalities:

$$\begin{aligned} p(a) &> p(b) + p(c) \\ p(c) + p(d) &> p(a) + p(b) \\ p(b) + p(e) &> p(a) + p(c) \end{aligned} \tag{2}$$

Adding,

$$p(a) + p(b) + p(c) + p(d) + p(e) > 2p(a) + 2p(b) + 2p(c)$$

so

$$p(d) + p(e) > p(a) + p(b) + p(c) \tag{3}$$

Suppose that we could find a qualitative probability $\succ'$ on X with the following properties: $\succ'$ satisfies the relations (1), $\succ'$ has an additive representation p', and there is no subset A of X such that $\{d,e\} \succeq A \succeq \{a,b,c\}$. Now consider the order $\succ$ such that $\{a,b,c\} \succ \{d,e\}$ and otherwise, for all subsets A and B such not both $A = \{a,b,c\}$ and $B = \{d,e\}$, $A \succ B$ iff $A \succ' B$. The relation $\succ$ provides the desired counterexample. It satisfies the relations (1), it is still a preference order, and the disjoint union property still holds. The second claim is true because since there are no sets in between $\{a,b,c\}$ and $\{d,e\}$, no transitive chains are broken. The third claim is true because since $\{a,b,c\}$ and $\{d,e\}$ partition X, there is no subset of X which is disjoint from the two of them to which the disjoint union property could apply. Finally, $\succ$ can have no additive representation. If it did, then rankings (1) implies that the representation satisfies (3). But since $\{a,b,c\} \succ \{d,e\}$, it must also be true that $p(a) + p(b) + p(c) > p(d) + p(e)$. Both inequalities cannot simultaneously hold.

Choose $0 < \epsilon < 1/3$ and consider the function p' such that

$$p'(a) = 4 - \epsilon \qquad p'(b) = 1 - \epsilon \qquad p'(c) = 3 - \epsilon \qquad p'(d) = 2 \qquad p'(e) = 6$$

This can be rescaled into a probability measure by dividing each term by $16 - 3\epsilon$. It is easy to check that the inequalities (2) are satisfied. Now $p(\{d,e\}) = 8$ and $p(\{a,b,c\}) = 8 - 3\epsilon$. Since $\epsilon < 1/3$, there can be a subset A of X in between $\{d,e\}$ and $a,b,c\}$ if and only if $p(A) = 8 - i\epsilon$ for some integer $i = 0,1,2$. The term 8 must come from adding up some of the numbers 1, 2, 3, 4 and 6. There are only two ways to make 8 out of these numbers. $8 = 1 + 3 + 4$ and $8 = 2 + 6$. Making 8 the first way requires that $A = \{a,b,c\}$, while making 8 the second way requires that $A = \{d,e\}$. Hence there can be no sets in between them. Take $\succ'$ to be the qualitative probability represented by p', and reverse the ranking of $\{d,e\}$ and $\{a,b,c\}$ to get $\succ$. $\square$

This example raises the question: What additional axioms are required to get an additive representation. Kraft et al. give a necessary and sufficient condition for a finite qualitative preference structure to have a probability representation, but it is not particularly intuitive so it will not be covered here. The following theorem applies to large qualitative probability structures, and is essentially due to Savage (1954). It requires an Archimedean assumption to fit into **R**. This assumption, not formally stated here, requires that if $A \succ \emptyset$, then there can be only a finite number of disjoint sets equivalent to A. A qualitative probability structure satisfying this requirement is called an *Archimedean qualitative probability structure.*

Theorem 9. *Suppose $(X, \mathcal{E}, \succ)$ is an Archimedean qualitative probability structure such that if $A \succ B$, there is a partition $C_1, \dots, C_n$ of X with each term in $\mathcal{E}$ such that $A \succ B \cup C_i$. Then $(X, \mathcal{E}, \succ)$ has a probability representation.*

The *fine* and *tight* conditions in Savage's axiom system imply the additional property. One problem with this theorem is that it requires X to be infinite.

Exercise 10. *Prove that the hypotheses of this theorem imply that X cannot be finite.*

The following theorem is due to Suppes (1969). It gives an additive representation for a finite qualitative probability structure — a qualitative probability structure for which X is finite — with the property that all *atoms* of $\mathcal{E}$ are equiprobable. An *atom* of $\mathcal{E}$ is a set $A \in \mathcal{E}$ such that $A \succ \emptyset$ and there is no $B \subset A$ such that $B \succ \emptyset$. The virtue of this theorem is not so much its applicability as its ease of demonstration. This is easy to prove, but illustrates the kinds of arguments measurement theorists make.

Theorem 10. *Suppose $(X, \mathcal{E}, \succ)$ is a finite qualitative probability structure such that if $A \succeq B$, there is a $C \in \mathcal{E}$ such that $A \sim B \cup C$. Then $\succ$ has a probability representation.*

Proof. First we throw away the null sets. Let N denote the union of all events Z such that $Z \sim \emptyset$. Then $N \sim \emptyset$. Define the following structure: $X' = X/N$, $\mathcal{E}' = \{A/N : A \in \mathcal{E}\}$. Finally, $A/N \succ' B/N$ iff $A \succ B$.

Now we show that all atoms are equiprobable. First observe that all atoms are disjoint. For if A and B are distinct atoms which are not disjoint, define $C = A \cap B$. If $C \sim \emptyset$, then $A/C \sim A$ and $A/C \subsetneq A$, so A is not an atom. If $C \succ \emptyset$, then A is also not an atom because $C \subsetneq A$.

Let A_1 denote a minimal atom, that is, an atom such that there is no set B such that $A_1 \succ B \succ \emptyset$. Let $\{A_1, \ldots, A_n\}$ denote the collection of distinct atoms equivalents to A_1. There are no additional atoms in $\mathcal{E}'$. To see this, suppose not. Then there is an atom A which is minimal among the class of atoms $A' \succ A_1$. That is, A is the least likely atom more likely than A_1. Write

$$B = A \cup \{A_2 \cup \cdots \cup A_n\}$$
$$C = A_1 \cup \{A_2 \cup \cdots \cup A_n\}$$

Since atoms are disjoint, it follows from 3. that $B \succ' C$. According to the hypotheses of the theorem, there is a set D such that $B \sim' C \cup D$. Without loss of generality, D can be assumed to be disjoint from C. Now D contains no minimal atoms (they are all in C) so $D \succeq' A$. But then

$$B \sim' C \cup D \succeq' C \cup A = B \cup A_1 \succ' B$$

which is a contradiction. In this statement, the first claim is by hypothesis. The second is a consequence of 3 (why?). The third follows from the definitions of B and C, and the last is a consequence of 3.

Now define the representation. For all $A \in \mathcal{E}$, $\mu(A) = \frac{1}{n}\#\{A_i : A_i \subset A/N\}$. First, check that μ is order preserving. This requires the following fact: If A is disjoint from both B and C, then $B \sim C$ iff $A \cup B \sim A \cup C$.

Exercise 11. *Prove this fact.*

Observe that for all A, $A \sim A/N$ and $\mu(A) = \mu(A/N)$.

If A and B both contain exactly m atoms, then $A \sim B$. Let $A' = A/N$ and $B' = B/N$. Then $A \sim A'$, $B \sim B'$, and A' and B' are each the union of m atoms. First suppose that A' and B' are disjoint. We prove this by induction on m. If $m = 1$ the claim is true because all atoms are equally likely. Suppose now the claim is true for $m - 1$. We want to prove it for m. Write $A' = \{A_1 \cup \cdots \cup A_{m-1}\} \cup A_m$, and $B' = \{B_1 \cup \cdots \cup B_{m-1}\} \cup B_m$, the union of their atoms. From the induction hypothesis, the two terms in brackets are equally likely. Since all sets are disjoint,

$$\begin{aligned} B' &= \{B_1 \cup \cdots \cup B_{m-1}\} \cup B_m \\ &\sim \{B_1 \cup \cdots \cup B_{m-1}\} \cup A_m \\ &\sim \{A_1 \cup \cdots \cup A_{m-1}\} \cup A_m \\ &= A' \end{aligned}$$

where each step is justified by 3. Thus $B' \sim A'$, so $B \sim A$.

Now suppose A' and B' contain k atoms in common. Let C denote the union of all atoms they contain in common. Then $A' = C \cup \{A_{k+1} \cup \cdots \cup A_m\}$ and $B' = C \cup \{B_{k+1} \cup \cdots \cup B_m\}$. The two sets in brackets are disjoint by construction. According to the previous argument, they are equally likely. It follows from 3 that $A' \sim B'$.

Next, if A contains m atoms and B contains $k < m$ atoms, then $A \succ B$. To see this, construct A' and B' as before. Write $A' = \{A_1 \cup \cdots \cup A_k\} \cup \{A_{k+1} \cup \cdots \cup A_m\}$, and $B' = \{B_1 \cup \cdots \cup B_k\}$, such that all atoms common to both A' and B' are among the first k on the list for A'. From the previous argument it follows that $\{A_1 \cup \cdots \cup A_k\} \sim \{B_1 \cup \cdots \cup B_k\}$. Then

$$\begin{aligned} A' &= \{A_1 \cup \cdots \cup A_k\} \cup \{A_{k+1} \cup \cdots \cup A_m\} \\ &\sim \{B_1 \cup \cdots \cup B_k\} \cup \{A_{k+1} \cup \cdots \cup A_m\} \\ &\succ \{B_1 \cup \cdots \cup B_k\} \\ &= B' \end{aligned}$$

where each step is justified by 3. It follows that $A' \succ B'$.

This exhausts all the cases. If $A \succ B$, then A must contain more atoms than B. If it contained fewer atoms than B, $B \succ A$. If A contained just as many atoms as B, then $A \sim B$. Since A contains more atoms than B, $\mu(A) > \mu(B)$. If $\mu(A) > \mu(B)$, then A contains more atoms than B, so $A \succ B$.

Next, we must check that μ is additive: That is, if A and B are disjoint, then $\mu(A \cup B) = \mu(A) + \mu(B)$. This is obvious from μ's definition. □

Necessary and sufficient conditions for a probability representation for a finite set X are simple. Let $\mathcal{Z}$ denote a collection of order-inequalities on $\mathcal{E}$:

$$A_1 \succeq (\succ) B_1 \qquad \cdots \qquad A_n \succeq (\succ) B_n$$

Let $L(x)$ and $R(x)$ denote the number of sets on the right and the left, respectively, containing the element $x \in X$.

It is surprisingly easy to prove for anyone who knows a little about convex sets, but I will not prove it here. See Scott (1964). The

Theorem 11. *Suppose* $(X, \mathcal{E}, \succ)$ *is a finite qualitative probability structure. It has a probability representation iff for every system of inequalities* $\mathcal{Z}$ *involving at least one strict inequality, it is not the case that for all* x*,* $L(x) = R(x)$*.*

Exercise 12. *Check to see that the condition of the theorem is violated in Example 6.*

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