An Anscombe-Aumann Model

April 24, 2014

1 Introduction

X Finite set of *prizes*.

S, $\mathcal{H}$ Set of *states*, set of *horse race events* $H \subset S$. If $H \in \mathcal{H}$, so is H^c. If H and J are in $\mathcal{H}$, so is $H \cup J$. S (and therefore $\emptyset$) is in $\mathcal{H}$.

$\mathcal{A}$ Set of *acts*. An act is a map from states to prizes. That is, $a : S \to X$. Furthermore, prizes depend on which horse-race event occurs. That is, for all $x \in X$, $a^{-1}(x) \in \mathcal{H}$. This requirement is called $\mathcal{H}$*-measurability*. Acts are simple.

$\mathcal{R}$ Set of *roulette events*. $\mathcal{R}$ contains all sub-intervals of $[0, 1)$, and the probability $P(r)$ of a sub-interval $R \in \mathcal{R}$ is given by the uniform distribution.

$\mathcal{L}$ Set of all *roulette lotteries*. A *roulette lottery* is a map from $r : [0, 1)$ to X which is $\mathcal{R}$-measurable. That is, $r^{-1}(x) \in \mathbf{R}$ for all $x \in X$. Lotteries are simple.

$\mathcal{G}$ The set of all gambles. $\mathcal{G} = \mathcal{A} \cup \mathcal{L}$.

Notice how this version of Anscombe-Aumann differs from that in Kreps. For Kreps (and in the original paper) a horse race is an $\mathcal{H}$-measurable function on S which maps states into lotteries rather than prizes.

A convenient way to write acts and lotteries is to list the prizes and the sets on which they are received. An act can be represented as a tuple $x_1H_1\cdots x_nH_nx_{n+1}$ where the $H_i \in \mathcal{H}$ are disjoint. The act pays off x_1 on H_1, etc., and x_{n+1} on $N_{n+1} = \left(\cup_{i=1}^n H_i\right)^c$. Similarly, a lottery can be represented as a tuple $x_1R_1\cdots x_nR_nx_{n+1}$ where the $R_i \in \mathcal{R}$ are disjoint elements of $\mathcal{R}$. A gamble $G \in \mathcal{G}$ is a tuple $x_1E_1\cdots x_nE_nx_{n+1}$ where the E_i are either all roulette events or all horse race events.

The constant acts can be identified with prizes. The

The decision maker has preferences on $\mathcal{G}$. The preference order satisfies the following properties.

Axiom 1 (Weak order). *$\succeq$ on $\mathcal{G}$ is complete and transitive.*

Axiom 2 (Domain). *There are prizes x^* and y^* such that $x^* \succ y^*$ and such that for all $H \in \mathcal{H}$, the act x^*Hy^* is in $\mathcal{A}$.*

Axiom 3 (Monotonicity). $x^* \succeq x^*Hy^* \succ y^*$

Axiom 4 (Independence for lotteries). *For lotteries $L, L', Q \in \mathcal{L}$, if $L \succ L'$ and $0 < \lambda \leq 1$, $\lambda L + (1-\lambda)Q \succ \lambda L' + (1-\lambda)Q$.*

Axiom 5 (Archimedean for lotteries). *For lotteries $L, L', L'' \in \mathcal{L}$, if $L \succ L' \succ L''$ there are $0 < \lambda, \mu < 1$ such that $\lambda L + (1-\lambda)L'' \succ L' \succ \mu L + (1-\mu)L''$.*

Axiom 6 (Additivity). *For all disjoint horse events H, H' there are disjoint roulette events R, R' such that $x^*Hy^* \sim x^*Ry^*$, $x^*H'y^* \sim x^*R'y^*$ and $x^*H \cup H'y^* \sim x^*R \cup R'y^*$.*

Axiom 7 (Probabilistic Beliefs). *If $x^*H_iy^* \sim x^*R_iy^*$ for all i, then for all prizes $x_1, \ldots, x_n$ and y, $x_1H_1\cdots x_nH_nx_{n+1} \sim x_1R_1\cdots x_nR_nx_{n+1}$.*

The representation theorem is:

Theorem 1. *Under the domain Axiom 2, the following two statements are equivalent:*

1. *There is a probability distribution Q on $\mathcal{H}$ and a utility function $U : X \to \mathbf{R}$ such that*

$$V(x_1E_1\cdots x_nE_nx_{n+1}) = \begin{cases} \sum_{i=1}^{n+1} P(E_i)U(x_i) & \text{for a roulette lottery,} \\ \sum_{i=1}^{n+1} Q(E_i)U(x_i) & \text{for a horse lottery.} \end{cases}$$

represents $\succeq$ on $\mathcal{G}$.

2. $\succeq$ on $\mathcal{G}$ satisfies Axioms 1, 3, 4, 5, 6 and 7.

Proof. That 1 implies 2 is obvious. I will show that the first claim is implied by the second. Axioms 1, 4 and 5 imply that $\succeq$ restricted to $\mathcal{L}$ has an expected utility representation with von-Neumann utility $U : X \to \mathbf{R}$. The Domain Axiom 2 implies that U is not constant, and, in particular, that $U(x^*) > U(y^*)$.

Define $Q(H) = P(R)$ for any R such that $x^*Hy^* \sim x^*Ry^*$. The existence of such an R is guaranteed by the additivity Axiom 6. If $P(R') > P(R)$, then from the EU representation for $\succeq$ on $\mathcal{L}$ we see that $x^*R'y^* \succ x^*Ry^*$. Similarly for $P(R') < P(R)$. Thus $Q(H)$ is well-defined. Additivity also implies that Q is additive. Suppose that H and H' are disjoint, and R and R' are such that the additivity axiom is satisfied. Then $x^*H \cup H'y^* \sim x^*R \cup R'y^*$, and so

$$Q(H \cup H') = P(R \cup R') = P(R) + P(R') = Q(H) + Q(H')$$

$\square$