Preferences for Intertemporal Choice

Larry Blume

Cornell University & The Santa Fe Institute & IHS

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Examples of Choice Problems

- Should I get a job after graduation and start making money now, or get a graduate degree and make more money starting two years from now?
- How much should I save out of each paycheck?
- I want to buy a house. How big a loan should I take?
- Food to cook this week: cheese, fish, pasta, steak, veggies. In which order should I eat them.
- The elevator problem.

Formalism

X_t Objects available at date t.
$X = X_0 \times X_1 \times \cdots$, set of bundles.

$\succ$ Preference relation on X.

Discounting

The “standard” preference order are those of the form

$$U(x_1, \ldots, x_T) = u_0(x_1) + u_1(x_1) + \cdots + u_T(x_T).$$

Such preferences are called additively separable.

Special case: stationary utility with a constant discount factor

$$u_t(x_t) = \beta^t u(x_t), \qquad \beta > 0$$

$$U(x_1, \ldots, x_T) = u(x_0) + \beta u(x_1) + \beta^2 u(x_2) + \cdots + \beta^T u(x_T).$$

Separable Preferences

Let $M \subset \{1, \ldots, T\}$ be a set of dates; $x = (x_M, , x_{\sim M})$.

Definition: Preferences are separable on M iff for all x_M, y_M, $x_{\sim M}$, $y_{\sim M}$, $(x_M, x_{\sim M}) \succ (y_M, x_{\sim M})$ if and only if $(x_M, y_{\sim M}) \succ (y_M, y_{\sim M})$.

Theorem: Suppose that preferences on X are represented by a utility function U. Then preferences are separable on M if and only if there is a utility functional $u : X_M \to \mathbf{R}$ and an aggregator $U^* : \mathbf{R} \times X_{\sim M} \to \mathbf{R}$ increasing in its first argument such that $U(x) = U^*\left(u(x_M), x_{\sim M}\right)$.

Proof

If U has this form, then $(x_M, x_{\sim M}) \succ (y_M, x_{\sim M})$ iff $U^*\left(u(x_M), x_{\sim M}\right) > U^*\left(u(y_M), x_{\sim M}\right)$. Since U^* is increasing in its first argument, $u(x_M) > u_(y_M)$. Thus for any other $y_{\sim M}$, $U^*\left(u(x_M), y_{\sim M}\right) > U^*\left(u(y_M), y_{\sim M}\right)$ and so $(x_M, y_{\sim M}) \succ (y_M, y_{\sim M})$.

Proof

If preferences are separable on M, pick $x'_{\sim M}$, and let $u(x_M) = U(x_M, x'_{\sim M})$. Define U^* such that $U^*\left(u(x_M), x_{\sim M}\right) = U(x_M, x_{\sim M})$. U^* will be well- defined iff there are no x_M, $x_{\sim M}$, y_M such that $u(x_M) = u(y_M)$ but $U(x_M, x_{\sim M}) \neq U(y_M, x_{\sim M})$. But we have $U(x_M, x'_{\sim M}) = U(y_M, x'_{\sim M})$, so separability implies that this holds for all $x_{\sim M}$.

If $u(x_M) > u(y_M)$ then

$$U^*\left(u(x_M), x'_{\sim M}\right) = U(x_M, x'_{\sim M}) > U(y_M, x'_{\sim M}) = U^*\left(u(y_M), x'_{\sim M}\right),$$

so separability implies that for all $x_{\sim M}$,

$$U^*\left(u(x_M), x_{\sim M}\right) = U(x_M, x_{\sim M}) > U(y_M, x_{\sim M}) = U^*\left(u(y_M), x_{\sim M}\right).$$

So U^* is increasing in its first argument.

The Definition Chains

Suppose that M and N are disjoint subsets of $\{1, \ldots, T\}$. Suppose that preferences are separable over both M and N. Then the utility function has the form $U^* \left(u_M(x_M), u_N(x_N), x_{\sim M \cup N} \right)$.

In fact, for disjoint $M_1, M_2, \ldots$. This can be shown by induction.

Separability and Indifference Curves

Suppose each $X_k = \mathbf{R}_+$ for $k = 1, 2, 3$, and fix x_1', and x_2'. The slope of the indifference curve in $X_1 \times X_2$ is independent of X_2. For $i = 1, 2$,

$$\frac{\partial U(x_1', x_2', x_3)}{\partial x_i} = U_1^* \left(u(x_1', x_2'), x_3 \right) \frac{\partial u}{\partial x_i},$$

and so

$$\frac{\partial U(x_1', x_2', x_3)/\partial x_1}{\partial U(x_1', x_2', x_3)/\partial x_2} = \frac{\partial u(x_1', x_2')/\partial x_1}{\partial u(x_1', x_2')/\partial x_2}$$

Separability and Optimization

$$\max U^*(u(x_M), x_{\sim M})$$
$$\text{s.t.} \quad p \cdot x_M + q \cdot x_{\sim M} \leq w$$
$$x_M \geq 0,\ x_{\sim M} \geq 0.$$

If the DM spends w_M on the goods in M and $w_{\sim M} = w - w_M$ on $x_{\sim M}$,

1. How should he allocate w_M in M?

$$v(p, w_M) = \max_{x_M} u(x_M)$$
$$\text{s.t.} \quad p \cdot x_M \leq w_M, \quad x_M \geq 0.$$

2. How should he choose w_M and $x_{\sim M}$?

$$\max_{w_M, x_{\sim M}} U^*\left(v(p, w_M), x_{\sim M}\right)$$
$$\text{s.t.} \quad w_M + q \cdot x_{\sim M} \leq w, \quad w_M, x_{\sim M} \geq 0.$$

Additive Separability

When is the aggregator $+$?

Definition: A factor t is essential if there exists quantities x_t, y_t and $x_{\sim\{t\}}$ such that $(x_t, x_{\sim\{t\}}) \succ (y_t, x_{\sim\{t\}})$.

Theorem: Suppose $\succ$ has a continuous utility representation on X and that there are at least three essential factors. Then $\succ$ has an additively separable representation iff each $\{t\}$ is separable.

If $\sum_t u_t(x_t)$ and $\sum_t v_t(x_t)$ both represent $\succ$, then there is an $a > 0$ and b_t such that $v_t(x) = au_t(x) + b_t$.

Separability and Expected Utility

Suppose $\mathcal{S} = \{1, 2, 3\}$, and $\mathcal{O} = \{a, b, c, d\}$. Consider two acts,

$$f(s) = \begin{cases} a & \text{if } s = 1, \\ b & \text{if } s = 2, \\ c & \text{if } s = 3, \end{cases} \qquad g(s) = \begin{cases} b & \text{if } s = 1, \\ a & \text{if } s = 2, \\ c & \text{if } s = 3. \end{cases}$$

$$f'(s) = \begin{cases} a & \text{if } s = 1, \\ b & \text{if } s = 2, \\ d & \text{if } s = 3, \end{cases} \qquad g'(s) = \begin{cases} b & \text{if } s = 1, \\ a & \text{if } s = 2, \\ d & \text{if } s = 3. \end{cases}$$

Suppose p is a probability distribution on $\mathcal{S}$ and $u : \mathcal{O} \to \mathbf{R}$ is a payoff function.

$$\begin{aligned} E_p u \circ f - E_p u \circ g &= p(1)\Big(u(o_1) - u(o_2)\Big) + p(2)\Big(u(o_2) - u(o_1)\Big) \\ &= E_p u \circ f' - E_p u \circ g' \end{aligned}$$

so $f \succ g$ iff $f' \succ g'$, and EU preferences are separable.

Suppose $\mathcal{P}$ is the set of probabilities such that $p(2) = p$ for a fixed $0 < p < 1/2$. Suppose $u(b) = 0$ and $u(d) > u(a) > 0 > u(c)$.

$$\min_{p \in \mathcal{P}} E_p u \circ f = (1-p)u(c) \qquad \min_{p \in \mathcal{P}} E_p u \circ g = (1-p)u(c)$$

so $f \sim g$, and

$$\min_{p \in \mathcal{P}} E_p u \circ f' = (1-p)u(a) \qquad \min_{p \in \mathcal{P}} E_p u \circ g' = pu(a)$$

so $f' \succ g'$.

Thus MMEU preferences are not separable.

Stationarity

Theorem: Suppose that $X_t = X_s$ for all s and t. Suppose that $\succ$ has an additively separable representation and for all $(x_1, \ldots, x_T)$ and y_1, $(x_1, \ldots, x_T) \succ (y_1, x_2, \ldots, x_T)$ iff $(x_2, \ldots, x_T, x_1) \succ (x_2, \ldots, x_T, y_1)$. Then we can take

$$u_1 = \cdots = u_T.$$

Dynamic Programming I

Suppose preferences are additively separable and consider the problem

$$\max_x \sum_t u_t(x_t)$$

$$\text{s.t.} \quad p \cdot x \leq w, \quad x \geq 0.$$

Solve the last period problem first, assuming the DM has wealth w_T.

$$v_T(p_T, w_T) = \max_{x_T} u_T(x_T)$$

$$\text{s.t.} \quad p_T \cdot x_T \leq w_T, \quad x_T \geq 0.$$

Now solve

$$v_{T-1}(p_{T-1}, w_{T-1}) = \max_{x_{T-1}} u_{T-1}(x_{T-1}) + v_T(p_T, w_T)$$

$$\text{s.t.} \quad p_{T-1} \cdot x_{T-1} \leq w_{T-1}, \quad w_T = w_{T-1} - p_{T-1} x_{T-1}, \quad x_{T-1} \geq 0.$$

And so forth.

Dynamic Programming II

This method is called backward induction.

$$v_{T-1}(p_{T-1}, w_{T-1}) = \max_{x_{T-1}} u_{T-1}(x_{T-1}) + v_T(p_T, w_T)$$

s.t. $\quad p_{T-1} \cdot x_{T-1} \leq w_{T-1}, \quad w_T = w_{T-1} - p_{T-1}x_{T-1}, \quad x_{T-1} \geq 0.$

v_T and v_{T-1} are the date T and date $T-1$ value function.

w_T and w_{T-1} are the date T and date $T-1$ state variables.

$w_T = w_{T-1} - p_{T-1}x_{T-1}$ is the equation of evolution or state equation.

Stationary Infinite Horizon Problems

Suppose the problem is stationary: $U(x) = \sum_{t=0}^{\infty} \beta^t u(x_t)$.
Suppose $p_t = \delta^t p$ for $0 < \delta < 1$.

The “successor budget constraint" is

$$\delta^t p x_t + \delta^{t+1} p x_{t+1} + \cdots \leq w_t.$$

Define $\tilde{w}_t = \delta^{-t} w_t$. Then

$$p x_t + \delta p x_{t+1} + \cdots \leq \tilde{w}_t.$$

Define $\tilde{w}_t$ as the state variable. The state evolution equation becomes

$$p x_t + \delta \tilde{w}_{t+1} = \tilde{w}_t.$$

The problem posed this way is stationary.

The Bellman Equation

$$v(\tilde{w}_t) = \max_x u(x) + \beta v(\tilde{w}_{t+1})$$

$$\text{s.t.} \quad p_t x_t + \delta \tilde{w}_{t+1} \leq \tilde{w}_t, \quad x_t \geq 0, \tilde{w}_{t+1} \geq 0.$$

Define the Bellman operator

$$Tv(w) = \max_x u(x) + \beta v(\tilde{w}_{t+1})$$

$$\text{s.t.} \quad p_t x_t + \delta w' \leq w, \quad x_t \geq 0, w' \geq 0.$$

Fact: For any v, the sequence $v, Tv, T^2 v, T^3 v, \ldots$ converges to the value function.

TIme Consistency

Suppose discounting is not necessarily geometric. Instead, the future k periods ahead is discounted at rate $d(k)$, so

$$U(x_1, \ldots) = \sum_{k=1}^{\infty} d(k) u(x_k).$$

where $d(1) = 1/(1 + \delta_1)$, and define inductively $d(k) = d(k-1) \cdot 1/(1 + \delta_k)$.

TIme Consistency

If $\delta_k \equiv \delta$ then if $x + y$ tomorrow is preferred to x today, then $x + y$ is preferred in period $t + k + 1$ to x in period $t + k$.

If $(c_t, c_{t+1}, \ldots)$ is preferred to $(c'_{t'}, c'_{t+1'}, \ldots)$ and $c_t = c'_{t'}$, then $(c_{t+1}, \ldots)$ is preferred to $(c'_{t'}, c'_{t+1} \ldots)$. The data is unclear on whether or not this happens in practice.

Hyperbolic discounting. If x at period t is preferred to $x + y$ at period $t + k$, then for all $h > 0$, x at period $t + h$ is preferred to $x + y$ at period $t + k + h$.