Static Decision Theory Under Certainty

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A set of objects $X$

An individual is asked to express preferences among the objects, or to make choices from subsets of $X$.

For $x, y \in X$ we can ask which, if either, is strictly preferred, that is, the best of the two.

If the subject says, “I prefer $x$ to $y$,” then we write $x \succ y$ and say, “$x$ is strictly preferred to $y$.”

The relation $\succ$ is a binary relation.

Example 1: $X = \{a, b, c\}$, $b \succ a$, $a \succ c$, and $b \succ c$. What if the subject also says $a \succ b$?
Axioms – properties that (arguably) all preference orders should satisfy.

**Asymmetry:** For all $x, y \in X$, if $x \succ y$ then $y \not\succ x$.

**Negative Transitivity:** For all $x, y, z \in X$, if $x \not\succ y$ and $y \not\succ z$ then $x \not\succ z$.

**Proposition:** The binary relation $\succ$ is negatively transitive iff $x \succ z$ implies that for all $y$, $y \succ z$ or $x \succ y$. 
Example 2: \( X = \{ a, b, c \}, \quad b \succ a, \quad a \succ c \) and \( b \preceq c \). Asymmetry and NT you also know how \( b \) and \( c \) must be ranked.

Definition: A binary relation \( \succ \) is called a (strict) preference relation if it is asymmetric and negatively transitive.

Is asymmetry a good normative or descriptive property? What about negative transitivity.
**Definition:** For $x, y \in X$,

- $x \succ y$ iff $y \nleq x$;
- $x \sim y$ iff $y \nleq x$ and $x \nleq y$.

Does the absence of strict preference in either direction require real indifference or could it permit non-comparability?

**Example:** $X = \{a, b, c\}$. Suppose $a$ is not ranked (by $\succ$) relative to either $b$ or $c$. If $\succ$ satisfies NT, then $b$ and $c$ are not ranked either.
Definition: The binary relation \( \succeq \) on \( X \) is **complete** if for all \( x, y \in X \), \( x \succeq y \) or \( y \succeq x \). \( \succeq \) is **transitive** iff for all \( x, y, z \in X \), \( x \succeq y \) and \( y \succeq z \) implies \( x \succeq z \).

Proposition: Let \( \succ \) be a binary relation on \( X \).
- \( \succ \) is asymmetric iff \( \succeq \) is complete.
- \( \succ \) is negatively transitive iff \( \succeq \) is transitive.
Proof: \[ \implies \]

- Asymmetry implies that for no pair \( x, y \in X \) is it true that both \( x \succ y \) and \( y \succ x \). Thus at least one of \( x \not\succ y \) and \( y \not\succ x \) must hold. So at least one of \( x \succeq y \) and \( y \succeq x \) is true. That is, \( \succeq \) is complete.

- If \( x \not\succ y \) and \( y \not\succ z \), then \( x \not\succ z \). By definition we have \( y \succeq x \) and \( z \succeq y \) implies \( z \succeq x \), so \( \succeq \) is transitive.

\[ \Longleftarrow \] will be on homework 1.
Proposition: If $\succ$ is a preference relation, then $\succ$ is transitive.

Is transitive a useful property?

- Normative property?
- The coffee cup example.
- Without transitivity, there may be no preference maximal object in a set of alternatives.
Suppose that \( X \) is finite. Let \( P^+(X) \) denote the set of all non-empty subsets of \( X \).

**Definition**: A choice function is a function \( c : P^+(X) \to P^+(X) \) such that for all \( A \in P^+(X) \), \( c(A) \subseteq A \).

\( c(A) \) is the set of objects “chosen” from \( A \).

Preference relations define choice functions.

**Definition**: For a preference relation \( \succ \) on \( X \), its choice function \( c_{\succ} : P^+(X) \to P^+(X) \) is

\[
c_{\succ}(A) = \{ x \in A : \text{ for all } y \in A, y \not\succ x \}.
\]
static decision theory

choice

Things to think about:

▶ Show that if \( x, y \in c_\succ (A) \), then \( x \sim y \).
▶ Show that for all \( A \in P^+(X) \), \( c_\succ (A) \neq \emptyset \).

The second item justifies the use of the phrase choice function to describe \( c_\succ \).
For every choice function $c$ is there a preference order $\succ$ such that $c = c_{\succ}$?

Clearly not:

Example: $X = \{a, b, c\}$.

- $c(\{a, b, c\}) = \{a\}$ and $c(\{a, b\}) = \{b\}$ violates asymmetry.
- $c(\{a, b\}) = \{a\}$ and $c(\{b, c\}) = \{b\}$ and $c(\{a, c\} = \{c\}$ violates negative transitivity.
Axiom \( \alpha \): If \( x \in B \subset A \) and \( x \in C(A) \), then \( x \in C(B) \).

Proposition: If \( \succ \) is a preference relation, then \( c_\succ \) satisfies axiom \( \alpha \).

Proof: Suppose there are sets \( A, B \in P^+(X) \) satisfying the hypotheses, that \( x \in c_\succ (A) \) and \( x \not\in c_\succ (B) \), Then there is a \( y \in B \) such that \( y \succ x \). Since \( B \subset A \), \( y \in A \) and so \( x \not\in c_\succ (A) \), contrary to our hypothesis.
Axiom $\beta$: If $x, y \in c(A)$, $A \subset B$ and $y \in c(B)$, then $x \in c(B)$.

Proposition: If $\succ$ is a preference relation, then $c_\succ$ satisfies axiom $\beta$.

Proof: Since $\in c_\succ(A)$ and $y \in A$, $y \not\succ x$. Since $y \in c_\succ(B)$, for all $z \in B$, $z \not\succ y$. Negative transitivity implies that for all $z \in B$, $z \not\succ x$. Thus $x \in c_\succ(B)$. 
Axioms $\alpha$ and $\beta$ characterize preference-based choice.

**Proposition:** If a choice function $c$ satisfies axioms $\alpha$ and $\beta$, then there is a preference relation $\succ$ such that $c = c \succ$.

**Proof:** Two steps

- Define a “revealed preference order” $\succ$ and show that it is a preference relation, i.e. asymmetric and negatively transitive.
- Show that $c = c \succ$. 

Define a preference order: \( x \succ y \) iff \( x \neq y \) and \( c(\{x, y\}) = \{x\} \). Notice that, by definition, \( x \not\succ x \).

- \( \succ \) is asymmetric.

  Suppose not. Suppose \( x \succ y \) and \( y \succ x \). Then \( c(\{x, y\}) = \{x\} \) and \( c(\{x, y\}) = \{y\} \). But both cannot be true.

- \( \succ \) is negatively transitive.

  Suppose that for some \( x, y, z \in X \), \( z \not\succ y \) and \( y \not\succ x \). Show that \( z \not\succ x \). That is, show that \( x \in c(\{x, z\}) \). It suffices to show \( x \in c(\{x, y, z\}) \), because then \( x \in c(\{x, z\}) \) follows from \( \alpha \).

  Suppose that \( x \notin c(\{x, y, z\}) \). Then one or both of \( y \) and \( z \) are in \( c(\{x, y, z\}) \) because \( c(\{x, y, z\}) \neq \emptyset \). We will show that neither of them can be in.
\begin{itemize}
\item $y \not\in c(\{x, y, z\})$.
\begin{itemize}
\item Suppose $y \in c(\{x, y, z\})$. Axiom $\alpha$ implies $y \in c(\{x, y\})$.
\item Since $y \not\succ x$, $x \in c(\{x, y\})$. Axiom $\beta$ implies $x \in c(\{x, y, z\})$.
\end{itemize}
\item $z \not\in c(\{x, y, z\})$.
\begin{itemize}
\item Suppose $z \in c(\{x, y, z\})$. Axiom $\alpha$ implies $z \in c(\{y, z\})$.
\item $z \not\succ y$ implies $y \in c(\{y, z\})$. Axiom $\beta$ implies $y \in c(\{x, y, z\})$.
\end{itemize}
\end{itemize}
“Revealed preferred to” $\succ$ is a preference relation. Now we have to show that for all $A \in P^+(A)$, $c(A) = c_\succ(A)$.

- Suppose $x \in c(A)$.
  
  $\alpha$ implies $x \in c(\{x, y\})$ for all $y \in A$. By definition, for all $y \in A$, $y \not\succ x$. Thus $x \in c_\succ(A)$.

- Suppose $\in c_\succ(A)$.
  
  Then for all $y \in A$, $y \not\succ x$, and so $x \in c(\{x, y\})$. Choose $z \in C(A)$. If $z \neq x$, axiom $\alpha$ implies $z \in c(\{x, z\})$, so $c(\{x, z\}) = \{x, z\}$. Axiom $\beta$ now implies $x \in C(A)$.

\[ \text{QED} \]
An alternative characterization of preference-based choice functions:

**Weak Axiom of Revealed Preference:** If \(x, y \in A \cap B\) and \(x \in c(A)\) and \(y \in c(B)\), then \(x \in c(B)\) and \(y \in c(A)\).

This axiom is called **Houthakker’s Axiom**, or WARP.

**Proposition:** \(c\) satisfies axioms \(\alpha\) and \(\beta\) iff it satisfies WARP.

**Proof:** ?
We have already dissed completeness of \( \succeq \).

**Definition:** \( \succ \) is a partial order iff it is asymmetric and transitive.

**Problem:** Characterize \( c_{\succ} \) for partial orders.

Axiom \( \alpha \) still holds, but \( \beta \) may fail. See homework 1.

Now we do not want to define indifference as before, since the usual definition expresses both indifference and non-comparability. One could define the pair \((\succ, \sim)\) and theorize about the pair.