Representations of Uncertainty

Goal: to find (and characterize) reasonable decision rule that deal with the Ellsberg paradox.
We've already seen one: a set $\mathcal{P}$ of probabilities. Recall that

$$\underline{E}_{\mathcal{P}}(u_a) = \inf_{\Pr \in \mathcal{P}} \{E_{\Pr}(u_a) : \Pr \in \mathcal{P}\}.$$

Thus, we get the rule MMEU (Maxmin Expected Utility):

$$a_1 \leq a_2 \text{ if } \underline{E}_{\mathcal{P}}(u_{a_1}) \leq \underline{E}_{\mathcal{P}}(u_{a_2}).$$

MMEU generalizes maximin (if $\mathcal{P}$ consists of all probability measures) and expected utility (if $\mathcal{P}$ consists of just one probability measure).

Characterizing EU

Recall the Anscombe-Aumann framework:

- the objects of choice are horse lotteries.
  - functions from state space S (assume finite) to simple probability distributions (i.e. distributions with finite support) over Z (prizes)

Here were the axioms that characterized expected utility maximization:

A1. $\succ$ is a preference relation on H (horse lotteries)

A2. (Continuity:) If $f \succ g \succ h$, then there exist $\alpha, \beta \in (0,1)$ such that $\alpha f + (1-\alpha)h \succ g \succ \beta f + (1-\beta)h$.

A3. (Independence:) If $f \succ g$, then for all h and $\alpha \in (0,1]$, $\alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h$.

If $X \subseteq S$, let $f_X g$ be the act that agrees with f on X and with g on X^c (the complement of X).

A4. (Monotonicity:) If p and q are probabilities on prizes and s and s' are non-null states, then $p_{\{s\}}f \succ q_{\{s\}}f$ iff $p_{\{s'\}}f \succ q_{\{s'\}}f$.

A5. (Nondegeneracy:) There exist f and g such that $f \succ g$.

Key result:

Theorem: (Anscombe-Aumann) If A1–A5 hold, then there exist a utility u on prizes and a probability Pr on states such that $\succ$ can be represented by expected utility.

- Can associate with each horse lottery h a random variable u_h:
  - $u_h(s)$ is the expected utility of the lottery $h(s)$ on prizes (i.e., $u_h(s) = \Sigma_{z \in Z}\, h(s)(z)u(z)$)
- $f \succ g$ iff $E_{\Pr}(u_f) > E_{\Pr}(u_g)$.

Moreover, Pr is unique and u is unique up to affine transformations.

Claim: A1 and A2 hold for MMEU, but A3 and A4 fail (see homework).

A3. (Independence:) If $f \succ g$, then for all h and $\alpha \in (0, 1]$, $\alpha f + (1 - \alpha)h \succ \alpha g + (1 - \alpha)h$.

Example: Suppose that

- $S = \{s_1, s_2\}$
- $\mathcal{P} = \{\Pr_1, \Pr_2\}$; $\Pr_1(s_1) = 1/3$, $\Pr_2(s_1) = 2/3$
- $f = (4.2, 4.2)$ (i.e. $f(s_1) = 4.2$; $f(s_2) = 4.2$), $g = (6, 3)$, $h = (3, 6)$.
- $\underline{E}(f) = 4.2$ and $\underline{E}(g) = 4$, so $f \succ g$.
- $f/2 + h/2 = (3.6, 5.1)$; $g/2 + h/2 = (4.5, 4.5)$
- $\underline{E}(f/2 + h/2) = 4.1$ and $\underline{E}(g/2 + h/2) = 4.5$, so $g/2 + h/2 \succ f/2 + h/2$.

Characterizing MMEU

[Gilboa and Schmeidler:] Independence doesn't hold; we replace it by:

A3′. (Certainty-Independence:) If $f \succ g$, h is a constant function, and $\alpha \in (0,1]$, then $\alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h$.

- A3 just says "if … then"; "iff" follows from other axioms.

Instead of A4, GS use:

A4′. (Monotonicity:) If $f(s) \succeq g(s)$ for all $s \in S$, then $f \succeq g$ (where $f \succeq g$ if not($g \succ f$)).

- This doesn't quite mean that f beats g at every state. Think of $f(s)$ as the constant horse lottery that returns $f(s)$ at every state. It means the constant $f(s)$ beats the constant $g(s)$.

One more property is needed:

A6. (Uncertainty Aversion:) If $\alpha \in (0,1)$ and $f \approx g$, then $\alpha f + (1-\alpha)g \succeq f$.

- For EU, A6 holds with $\approx$ (follows from A1–A3).
- Can have $\alpha f + (1 - \alpha g) \succ f$ with MMEU
  - Consider previous example: $g = (6,3)$, $h = (3,6)$. Then $g \approx h$, but $g/2 + h/2 \succ g$
- A6 models hedging.

Theorem: (Gilboa-Schmeidler) If A1, A2, A3′, A4′, A5, and A6 hold, then there exist a utility u on prizes and a closed convex set $\mathcal{P}$ of probability measures on states such that $\succ$ can be represented by MMEU.

- $f \succ g$ iff $\underline{E}_{\mathcal{P}}(u_f) > \underline{E}_{\mathcal{P}}(u_g)$

Moreover, $\mathcal{P}$ is unique and u is unique up to affine transformations.

- All you really need are the extreme points in $\mathcal{P}$; requiring that $\mathcal{P}$ be closed and convex makes it unique.

Other Representations of Uncertainty

Why is probability the "right" way to represent uncertainty?

- It's not so good at representing ignorance.
- or extremely unlikely events.

Many alternatives considered in the literature:

- sets of probabilities
- non-additive probabilities
- belief functions
- lexicographic probabilities
- possibility measures
- ranking functions
- plausibility measures
- ...

Some of these approaches are closely related. We'll focus on sets of probabilities, non-additive probabilities, and belief functions.

- If want more, take CS 6766!

Non-additive probabilities

A *non-additive probability* [Choquet, Schmeidler] ν on S is a function mapping subsets of S to $[0,1]$ such that

N1. $\nu(\emptyset) = 0$

N2. $\nu(S) = 1$

N3. If $E \subseteq F$, then $\nu(E) \leq \nu(F)$.

These constraints are pretty minimal. For example, suppose $S = \{s_1, s_2\}$ and

- $\nu_\alpha(\emptyset) = 0$
- $\nu_\alpha(s_1) = \nu_\alpha(s_2) = \alpha$
- $\nu(S) = 1$.

Then ν_α is a nonadditive probability for each $\alpha \in [0,1]$.

We may want more constraints ...

Expectation with respect to a nonadditive probability

Suppose that f is a random variable with finite range.

- Suppose that the values of f are $x_1 < \ldots < x_n$.

Then the expectation of f with respect to ν is defined as follows [Choquet]:

$$E_\nu(f) = x_1 + (x_2 - x_1)\nu(f > x_1) + \cdots + (x_n - x_{n-1})\nu(f > x_{n-1}).$$

Why is this the right definition of expectation?

- Some good news: it coincides with the standard definition if ν is a probability measure.

But why not use the more obvious generalization of probabilistic expectation?

$$E'_\nu(f) = \sum_{s \in S} \nu(s) f(s)$$

Stay tuned ...

Nonadditive Expected Utility

Nonadditive expected utility rule:

- Given a utility function u on prizes and a nonadditive probability ν on states, then

$$f \succ g \text{ iff } E_\nu(u_f) > E_\nu(u_g)$$

Comonotonic Independence

Acts f and g are *comonotonic* if there do not exist states s and t such that

$$f(s) \succ f(t) \text{ and } g(t) \succ g(s)$$

- f and g are comonotonic if you can't be happier to be in state s than state t when doing f and be happier to be in state t than state s when doing g.
- If h is a constant act, then f and h are comonotonic for all acts f (since we never have $h(s) \succ h(t)$).

A3$''$. (Comonotonic Independence:) If f and h and g and h are both comonotonic and $f \succ g$, then for all $\alpha \in (0, 1]$, $\alpha f + (1 - \alpha)h \succ \alpha g + (1 - \alpha)h$.

Idea: comonotonic independence tries to avoid the kind of application of independence that gives Ellsberg's paradox.

- Note: A3$''$ is stronger than A3$'$.

Representation Theorem

Theorem: (Schmeidler) If A1, A2, A3$''$, A4$'$, and A5 hold, then there exist a utility u on prizes and a nonadditive probability ν on states such that $\succ$ can be represented by NEU.

- $f \succ g$ iff $E_\nu(u_f) > E_\nu(u_g)$

Moreover, ν is unique and u is unique up to affine transformations.

- Moving from additive probability to nonadditive probability results in weakening independence to comonotonic independence.

Nonadditive Probability and Sets of Probabilities

Where might a nonadditive probability come from? One case:

- Given a set $\mathcal{P}$ of probabilities, define $\mathcal{P}_*$ to be the lower probability of $\mathcal{P}$ and $\mathcal{P}^*$ to be the upper probability:

$$\begin{array}{c} \mathcal{P}_*(X) = \inf_{\Pr \in \mathcal{P}} \Pr(X) \\ \mathcal{P}^*(X) = \sup_{\Pr \in \mathcal{P}} \Pr(X). \end{array}$$

- $\mathcal{P}_*$ and $\mathcal{P}^*$ are both nonadditive probabilities; moreover

$$\mathcal{P}^*(A) = 1 - \mathcal{P}_*(A^c)$$

Is every nonadditive probability $\mathcal{P}_*$ (or $\mathcal{P}^*$) for some set $\mathcal{P}$ of probabilities?

- Simple counterexample: Let $S = \{s_1, s_2\}$.
  If $\nu_1(s_1) = 2/3$, $\nu_1(s_2) = 2/3$, then $\nu_1 \neq \mathcal{P}_*$.
  If $\nu_2(s_1) = 1/3$, $\nu_2(s_2) = 1/3$, then $\nu_2 \neq \mathcal{P}^*$.

Some properties of $\mathcal{P}_*$ and $\mathcal{P}^*$:

$$\mathcal{P}^*(A) + \mathcal{P}^*(B) \geq \mathcal{P}^*(A \cup B) \text{ if } A \cap B = \emptyset$$
$$\mathcal{P}_*(A) + \mathcal{P}_*(B) \leq \mathcal{P}_*(A \cup B) \text{ if } A \cap B = \emptyset$$
$$\mathcal{P}_*(A) + \mathcal{P}_*(B) \leq \mathcal{P}_*(A \cap B) + \mathcal{P}^*(A \cup B)$$
$$\leq \mathcal{P}^*(A) + \mathcal{P}^*(B)$$

(There are other properties too.)

Motivating the funny notion of expectation

Suppose that

- $S = \{s_1, s_2\}$
- $\mathcal{P} = \{\Pr_1, \Pr_2\}$,
- $\Pr_1(s_1) = 1$, $\Pr_2(s_2) = 1$.

Thus,“

- $\mathcal{P}_*(s_1) = \mathcal{P}_*(s_2) = 0$, $\mathcal{P}^*(s_1) = \mathcal{P}^*(s_2) = 1$.

Let f be the constant function 2.

- Using the “obvious” definition of expectation,
  - $E'_{\mathcal{P}_*}(f) = 2\Pr_*(s_1) + 2\Pr_*(s_2) = 0$.
  - $E'_{\mathcal{P}^*}(f) = 2\Pr^*(s_1) + 2\Pr^*(s_2) = 4$.
- The good news: $E_{\mathcal{P}_*}(f) = E_{\mathcal{P}^*}(f) = 2$.

E' given “wrong” answer; E gives the right answer.

- The expected value of the constant function 2 should be 2!