Static Decision Theory Under Certainty

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A set of objects $X$

An individual is asked to express preferences among the objects, or to make choices from subsets of $X$.

For $x, y \in X$ we can ask which, if either, is strictly preferred, that is, the best of the two.

If the subject says, “I prefer $x$ to $y$,” then we write $x \succ y$ and say, “$x$ is strictly preferred to $y$.”

The relation $\succ$ is a binary relation.

Example 1: Let $X = \{a, b, c\}$, $b \succ a$, $a \succ c$, and $b \succ c$. What if the subject also says $a \succ b$?
Axioms – properties that (arguably) all preference orders should satisfy.

**Asymmetry:** For all $x, y \in X$, if $x \succ y$ then $y \prec x$.

**Negative Transitivity:** For all $x, y, z \in X$, if $x \not\succ y$ and $y \not\succ z$ then $x \not\succ z$.

**Proposition:** The binary relation $\succ$ is negatively transitive iff $x \succ z$ implies that for all $y$, $y \succ z$ or $x \succ y$. 
Example 2: $X = \{a, b, c\}$, $b > a$, $a > c$ and $b \not> c$. Asymmetry and NT you also know how $b$ and $c$ must be ranked.

**Definition:** A binary relation $>$ is called a (strict) preference relation if it is asymmetric and negatively transitive.

Is asymmetry a good normative or descriptive property? What about negative transitivity.
Definition: For $x, y \in X$,

- $x > y$ iff $y \not\geq x$;
- $x \sim y$ iff $y \not\succ x$ and $x \not\prec y$.

Does the absence of strict preference in either direction require real indifference or could it permit non-comparability?

Example: $X = \{a, b, c\}$. Suppose $a$ is not ranked (by $>$) relative to either $b$ or $c$. If $>$ satisfies NT, then $b$ and $c$ are not ranked either.
Definition: The binary relation $\succeq$ on $X$ is complete if for all $x, y \in X$, $x \succeq y$ or $y \succeq x$. $\succeq$ is transitive iff for all $x, y, z \in X$, $x \succeq y$ and $y \succeq z$ implies $x \succeq z$.

Proposition: Let $>$ be a binary relation on $X$.
- $>$ is asymmetric iff $\succeq$ is complete.
- $>$ is negatively transitive iff $\succeq$ is transitive.
Proof: $\implies$

- Asymmetry implies that for no pair $x, y \in X$ is it true that both $x \succ y$ and $y \succ x$. Thus at least one of $x \nless y$ and $y \nless x$ must hold. So at least one of $x \preceq y$ and $y \preceq x$ is true. That is, $\preceq$ is complete.

- If $x \nless y$ and $y \nless z$, then $x \nless z$. By definition we have $y \succeq x$ and $z \succeq y$ implies $z \succeq x$, so $\succeq$ is transitive.

$\Longleftarrow$ will be on homework 1.
Proposition: If $\succ$ is a preference relation, then $\succ$ is transitive.

Is transitive a useful property?

- Normative property?
- The coffee cup example.
- Without transitivity, there may be no preference maximal object in a set of alternatives.
Suppose that $X$ is finite. Let $P^+(X)$ denote the set of all non-empty subsets of $X$.

**Definition:** A choice function is a function $c : P^+(X) \rightarrow P^+(X)$ such that for all $A \in P^+(X)$, $c(A) \subseteq A$.

$c(A)$ is the set of objects “chosen” from $A$.

Preference relations define choice functions.

**Definition:** For a preference relation $>$ on $X$, its choice function $c_\succ : P^+(X) \rightarrow P^+(X)$ is

$$c_\succ(A) = \{x \in A : \text{ for all } y \in A, y \not\succ x\}.$$
Things to think about:

- Show that if $x, y \in c_>(A)$, then $x \sim y$.
- Show that for all $A \in P^+(X)$, $c_>(A) \neq \emptyset$.

The second item justifies the use of the phrase choice function to describe $c_>$. 
For every choice function $c$ is there a preference order $\succ$ such that $c = c_\succ$?

Clearly not:

**Example:** $X = \{a, b, c\}$.

- $c(\{a, b, c\}) = \{a\}$ and $c(\{a, b\}) = \{b\}$ violates asymmetry.
- $c(\{a, b\}) = \{a\}$ and $c(\{b, c\}) = \{b\}$ and $c(\{a, c\} = \{c\}$ violates negative transitivity.
Axiom $\alpha$: If $x \in B \subset A$ and $x \in C(A)$, then $x \in C(B)$.

Proposition: If $\succ$ is a preference relation, then $c_\succ$ satisfies axiom $\alpha$.

Proof: Suppose there are sets $A, B \in P^+(X)$ satisfying the hypotheses, that $x \in c_\succ(A)$ and $x \notin c_\succ(B)$, Then there is a $y \in B$ such that $y \succ x$. Since $B \subset A$, $y \in A$ and so $x \notin c_\succ(A)$, contrary to our hypothesis.
**Axiom β:** If \( x, y \in c(A) \), \( A \subset B \) and \( y \in c(B) \), then \( x \in c(B) \).

**Proposition:** If \( \succ \) is a preference relation, then \( c_\succ \) satisfies axiom \( \beta \).

**Proof:** Since \( \in c_\succ(A) \) and \( y \in A \), \( y \not\succ x \). Since \( y \in c_\succ(B) \), for all \( z \in B \), \( z \not\succ y \). Negative transitivity implies that for all \( z \in B \), \( z \not\succ x \). Thus \( x \in c_\succ(B) \).
Axioms $\alpha$ and $\beta$ characterize preference-based choice.

**Proposition:** If a choice function $c$ satisfies axioms $\alpha$ and $\beta$, then there is a preference relation $\succ$ such that $c = c_{\succ}$.

**Proof:** Two steps

- Define a “revealed preference order” $\succ$ and show that it is a preference relation, i.e. asymmetric and negatively transitive.
- Show that $c = c_{\succ}$. 
Define a preference order: $x > y$ iff $x \neq y$ and $c(\{x, y\}) = \{x\}$. Notice that, by definition, $x \not> x$.

- $>$ is asymmetric.

Suppose not. Suppose $x > y$ and $y > x$. Then $c(\{x, y\}) = \{x\}$ and $c(\{x, y\}) = \{y\}$. But both cannot be true.

- $>$ is negatively transitive.

Suppose that for some $x, y, z \in X$, $z \not> y$ and $y \not> x$. Show that $z \not> x$. That is, show that $x \in c(\{x, z\})$. It suffices to show $x \in c(\{x, y, z\})$, because then $x \in c(\{x, z\})$ follows from $\alpha$.

Suppose that $x \notin c(\{x, y, z\})$. Then one or both of $y$ and $z$ are in $c(\{x, y, z\})$ because $c(\{x, y, z\}) \neq \emptyset$. We will show that neither of them can be in.
static decision theory proof

- \( y \notin c(\{x, y, z\}) \).

  Suppose \( y \in c(\{x, y, z\}) \). Axiom \( \alpha \) implies \( y \in c(\{x, y\}) \). Since \( y \not\succ x \), \( x \in c(\{x, y\}) \). Axiom \( \beta \) implies \( x \in c(\{x, y, z\}) \).

- \( z \notin c(\{x, y, z\}) \).

  Suppose \( z \in c(\{x, y, z\}) \). Axiom \( \alpha \) implies \( z \in c(\{y, z\}) \). \( z \not\succ y \) implies \( y \in c(\{y, z\}) \). Axiom \( \beta \) implies \( y \in c(\{x, y, z\}) \).
“Revealed preferred to” $\succ$ is a preference relation. Now we have to show that for all $A \in P^+(A)$, $c(A) = c_>(A)$.

- Suppose $x \in c(A)$.
  
  $\alpha$ implies $x \in c(\{x, y\})$ for all $y \in A$. By definition, for all $y \in A$, $y \not\succ x$. Thus $x \in c_>(A)$.

- Suppose $x \in c_>(A)$.
  
  Then for all $y \in A$, $y \not\succ x$, and so $x \in c(\{x, y\})$. Choose $z \in C(A)$. If $z \neq x$, axiom $\alpha$ implies $z \in c(\{x, z\})$, so $c(\{x, z\}) = \{x, z\}$. Axiom $\beta$ now implies $x \in C(A)$.

$\text{QED}$
An alternative characterization of preference-based choice functions:

**Weak Axiom of Revealed Preference:** If \( x, y \in A \cap B \) and \( x \in c(A) \) and \( y \in c(B) \), then \( x \in c(B) \) and \( y \in c(A) \).

This axiom is called **Houthakker’s Axiom**, or **WARP**.

**Proposition:** \( c \) satisfies axioms \( \alpha \) and \( \beta \) iff it satisfies WARP.

**Proof:** ?
We have already dissed completeness of $\succeq$.

**Definition:** $>$ is a partial order iff it is asymmetric and transitive.

**Problem:** Characterize $c_>$ for partial orders.

Axiom $\alpha$ still holds, but $\beta$ may feel. See homework 1.

Now we do not want to define indifference as before, since the usual definition expresses both indifference and non-comparability. One could define the pair $(>\,\sim)$ and theorize about the pair.