Problem Set 3:

Handed out: Oct. 28, 2010. Due: Nov. 11, 2010

For all of the problems on this problem set you can assume that the axioms we used in proving the Von Neumann-Morgenstern Theorem (Theorem 5.4 in Kreps) are satisfied. Note that by Lemma 5.7 of Kreps, these axioms imply that there are best and worst elements of the set of prizes X. You can use this lemma without proof.

1. Suppose that the set X of prizes in $\{30, 50, 70\}$. Gamble I delivers these prizes with probabilities 1/4, 1/4 and 1/2, respectively. Gamble 2 pays off 30 with probability 1. For the prizes $x \in \{30, 50, 70\}$, the payoff function has the form $u(x) = x - bx^2$.
   (a) Which gamble is preferred when $b = 0.005$?
   (b) Which is preferred when $b = 0.01$?
   (c) For which b is the decision maker indifferent between the two gambles?
2. Consider a finite set of prizes X and probabilities P on them. Suppose that an individual's preferences $\succ$ on P have an expected utility representation with utility function on prizes $u : X \to \mathbf{R}$. Show that $\succ$ satisfies the independence axiom.
3. Consider a finite set of prizes X and probabilities P on them. Suppose that an expected utility maximizer's preferences $\succ$ on P have an expected utility representation with utility function on prizes $u : X \to \mathbf{R}$. Suppose that $v(\cdot) = au(\cdot) + b$ for real numbers $a > 0$ and b. Show that v also represents $\succ$.
4. GRAD: State and prove the converse to the previous question's claim. That is, show that if $\succ$ has an expected utility representation using utility function u, and another expected utility representation using utility function v, then $v(\cdot) = au(\cdot) + b$ for real numbers $a > 0$ and b.
5. An expected utility-maximizing individual with wealth w will lose $a < w$ with probability p, and otherwise have no loss. The individual can

buy insurance at r per unit. That is, if she pays rx, she will receive x dollars in the event of a loss, and 0 otherwise.

(a) If she buys x units of insurance, what will be her wealth in the event of a loss? (Don't forget to account for the premium.)

(b) If she buys x units of insurance, what will be her wealth in if no loss occurs?

(c) What value x^* of x will make these two dollar amounts equal?

(d) Suppose that the payoff function $u(\cdot)$ has $u'(z) > 0$ and $u''(z) < 0$ for all $z > 0$. For what price r will she demand $x = x^*$ units of insurance. Such insurance is said to be actuarially fair. [Hint: Think about first-order conditions for the problem of choosing x to maximize expected utility.]

6. Consider again the portfolio choice problem discussed in class. Recall that you have initial wealth w_0, some of which you want to keep in cash and the rest of which you want to invest. Suppose that you keep m in cash and invest x, so that $w_0 = m + x$. (In the notes, instead px was used instead of x — this corresponds to buying x units of the asset at a price of p per unit; in this case, $w_0 = m + px$. It's fine if you use px instead of x. Either way, the total amount of money + risky asset should be w_0.(A portfolio containing m units of money and x units of assets has a random payoff $\tilde{y} = m + x\tilde{r}$, where $\tilde{r}$ is normally distributed with mean r and variance σ^2.

(a) What is the distribution of $\tilde{y}$?

(b) Suppose the investor is an expected utility maximizer with payoff function

$$u(y) = -e^{-\lambda y}.$$

The expected utility is $E - e^{-\lambda \tilde{y}}$. Express the expected utility of the portfolio (m, x) as a function of m, x, r and σ^2.

(c) Compute the optimal portfolio as a function of the asset return distribution r and σ^2.

7. Suppose an expected utility-maximizing investor has a payoff function of the form $u(x) = \gamma^{-1}x^{-\gamma}$ with $\gamma \leq 1$. He holds a portfolio which pays off either 85 or 115 with equal probability. If he sells the asset at price p, he will have a sure return of p.

(a) At what price p^* will the investor just be willing to sell the asset, for $\gamma = 1, 1/2, -1$.

(b) For arbitrary $\gamma \leq 1$, how does the sale price p^* vary with γ?