Experiment

Did the subjects make choices “as if” they had a preference relation $\succ$ over bundles of (IC, HB)? If so, could we infer $\succ$ and predict future choices or offer advice about choices?

In situation 2 the amount of money was $3.00 and the prices were $p_{HB} = .50$ and $p_{IC} = 1.00$; in situation 5 the amount of money was $3.60 and the prices were $p_{HB} = .60$ and $p_{IC} = 1.20$. The affordable set was the same in these two cases. So if the framing of the question doesn’t matter would expect the same choice in 2 as in 5.

22% of the subjects did not make the same choice in these two situations.

So we observe $x \succ y$ and $y \succ x$ for these people.
Lets look at situation 4 versus situation 1. In situation 4 the amount of money was $4.20 and the prices were $p_{HB} = .80$ and $p_{IC} = 1.20$; in situation 1 the amount of money was $3.60$ and the prices were $p_{HB} = .40$ and $p_{IC} = 1.60$. The affordable sets in these two cases are graphed below.

If we observe choices $x$ at 4 and $y$ at 1 then we have $x \succ y$ and $y \succ x$. No one made choices like this.
Static Decision Theory Under Certainty

A set of objects $X$.

An individual is asked to express his preferences among these objects or is asked to make choices from subsets of $X$.

For $x, y \in X$ we can ask which, if either, is strictly preferred.

- If the individual says $x$ is strictly better than $y$ we write $x \succ y$, read as $x$ is strictly preferred to $y$.

- $\succ$ is a binary relation on $X$.

Example 1: $X = \{a, b, p\}$, $b \succ a$, $a \succ p$ and $b \succ p$.

What if the answers also included $a \succ b$?
Axioms

**Asymmetry:** For any \( x, y \in X \) if \( x \succ y \) then not\([y \succ x]\).

**Negative Transitivity:** For any \( x, y, z \in X \) if not\([x \succ y]\) and not\([y \succ z]\) then not\([x \succ z]\).

**Proposition.** The binary relation \( \succ \) is negatively transitive iff \( x \succ z \) implies that, for all \( y \in X \), \( x \succ y \) or \( y \succ z \).

Example 2: \( X = \{a, b, c\} \), \( b \succ a \), \( a \succ c \) and \( b \nmid c \). If we have asymmetry and NT you also know how \( b \) and \( c \) must be ranked.

**Definition.** A binary relation \( \succ \) is called a (strict) preference relation if it is asymmetric and negatively transitive.

Is Asymmetry a good normative or descriptive property? What about NT?
Weak Preference

**Definition.** For $x, y \in X$:

1. $x$ is *weakly preferred* to $y$, $x \succeq y$, if not $[y \succ x]$.
2. $x$ is *indifferent* to $y$, $x \sim y$, if not $[x \succ y]$ and not $[y \succ x]$.

Does the absence of strict preference in either direction require real indifference or could it permit non-comparability?

**Example.** $X = \{a, b, c\}$. Suppose $a$ is not ranked (by $\succ$) relative to either $b$ or $c$. If $\succ$ satisfies NT, then $b$ and $c$ are not ranked either.

An interesting alternative would be to ask about $\succ$ and $\sim$ separately. Then define $x \succeq y$ as either $x \succ y$ or $x \sim y$. This permits the possibility that $x$ and $y$ are not comparable.
Definition. The binary relation $\succeq$ on $X$ is complete if for all $x, y \in X$, $x \succeq y$, $y \succeq x$ or both. It is transitive if $x \succeq y$ and $y \succeq z$ implies $x \succeq z$.

Proposition. Let $\succ$ be a binary relation on $X$.

1. $\succ$ is asymmetric iff $\succeq$ is complete.

2. $\succ$ is negatively transitive iff $\succeq$ is transitive.

Proof of $\Rightarrow$

1. By asymmetry of $\succ$ there is no pair $x, y \in X$ such that both $x \succ y$ and $y \succ x$. So at least one of not[$x \succ y$] and not[$y \succ x$] is true. Thus for any $x, y \in X$ either $y \succeq x$ or $x \succeq y$ or both. This is completeness.

2. Using the definition of $\succeq$, negative transitivity of $\succ$ is: for any $x, y, z \in X$, $y \succeq x$ and $z \succeq y$ implies $z \succeq x$. This is transitivity.

$\Leftarrow$ will be on homework 1.
Transitivity

Why do we care about transitivity?

**Remark:** If \( \succ \) is a preference relation then \( \succ \) is transitive.

Normative property?

Important for choice.

**Example.** \( X = \{a, b, p\} \). Consider a sequence of choices from among pairs.

1. \( \{a, b\} \), \( a \succ b \) and \( a \) is chosen.
2. \( \{a, p\} \), \( p \succ a \) and \( p \) is chosen.
3. \( \{p, b\} \), \( b \succ p \) and \( b \) is chosen.
4. \( \{a, b\} \ldots \)

Without transitivity can get cycles.

**Remark:** If \( \succ \) is a preference relation then \( \succ \) is acyclic, i.e. \( [x_1 \succ x_2 \succ \ldots x_{n-1} \succ x_n] \Rightarrow [x_1 \neq x_n] \).
Choice

Extend binary comparisons to choice over a set of more objects.

A finite set of objects $X$. Let $P(X)$ be the set of all non-empty subsets of $X$.

**Definition.** For $\succ$ a preference relation on $X$ define $c(\cdot, \succ)$ by, for $A \in P(X)$,

$$c(A, \succ) = \{x \in A : \text{for all } y \in A, y \not\succ x\}.$$ 

Interpretation: $c(A, \succ)$ is the set of alternatives chosen from $A$ by a decision maker with preferences $\succ$.

**Remark:** If $x, y \in c(A, \succ)$ then $x \sim y$.

**Proposition.** For $\succ$ a preference relation on a finite set $X$,

$$c(\cdot, \succ) : P(X) \to P(X).$$
What else do we know about $c(\cdot, A)$?

Consider general choice functions and ask what is special about $c(\cdot, A)$.

**Definition.** A choice function for $X$ is a function $c : P(X) \rightarrow P(X)$ such that for all $A \in P(X)$, $c(A) \subset A$.

Clearly, $c(\cdot, \succ)$ is a choice function.

Can any choice function be generated by some preference relation $\succ$? No.

**Example.** $X = \{a, b, c\}$.

1. $c(\{a, b, c\}) = \{a\}$ and $c(\{a, b\}) = \{b\} \Rightarrow$ a violation of asymmetry.

2. $c(\{a, b\}) = \{a, b\}$ and $c(\{a, b, c\}) = \{b\} \Rightarrow$ a violation of NT.
Axioms

Sen’s $\alpha$. If $x \in B \subset A$ and $x \in C(A)$, then $x \in C(B)$.

Independence of Irrelevant Alternatives.

**Proposition.** If $\succ$ is a preference relation then $c(\cdot, \succ)$ satisfies Sen’s $\alpha$.

**Proof.** Suppose there are sets $A, B \in P(X)$ with $B \subset A$, $x \in c(A, \succ)$ and $x \notin c(B, \succ)$. Then there is a $y \in B$ such that $y \succ x$. Since $B \subset A$ we have $y \in A$ and $y \succ x$. Thus $x \notin c(A, \succ)$. A contradiction.
Sen’s $\beta$. If $x, y \in c(A), A \subset B$ and $y \in c(B)$ then $x \in C(B)$.

**Proposition.** If $\succ$ is a preference relation then $c(\cdot, \succ)$ satisfies Sen’s $\beta$.

**Proof.** Since $x \in c(A, \succ)$ and $y \in A$ we have $y \not\succ x$. By definition, $y \in c(B, \succ)$ implies that for all $z \in B, z \not\succ y$. By negative transitivity, $y \not\succ x$ and $z \not\succ y$ implies $z \not\succ x$. Since $x \in B$ and this holds for all $z \in B$ we have $x \in c(B, \succ)$. 
Are there any other restrictions on $c(\cdot, \succ)$ that follow from $\succ$ being a preference relation? No.

**Proposition.** If a choice function $c$ satisfies Sen’s $\alpha$ and $\beta$, then there is a preference relation $\succ$ such that $c(\cdot) = c(\cdot, \succ)$.

Define the “revealed preference” relation $\succ$ by

$$x \succ y \text{ if } x \neq y \text{ and } c(\{x, y\}) = \{x\}.$$  

To prove the proposition we need to show that $\succ$ is a preference relation and that $c(\cdot) = c(\cdot, \succ)$. 
Proof

To show that $\succ$ is a preference relation we need to show that it is asymmetric and negatively transitive.

1. Asymmetry. Suppose for some $x$ and $y$, that $x \succ y$ and $y \succ x$. Then $c(\{x, y\}) = \{x\}$ and $c(\{x, y\}) = \{y\}$. A contradiction.

2. Negative Transitivity. Suppose that for some $x, y, z \in X$ we have $z \not\succ y$ and $y \not\succ x$. We need to show that $z \not\succ x$. This is $x \in c(\{x, z\})$. By Sen’s $\alpha$, showing that $x \in c(\{x, y, z\})$ is sufficient. Suppose $x \not\in c(\{x, y, z\})$. Then at least one of $y$ and $z$ are in $c(\{x, y, z\})$.

Suppose $y \in c(\{x, y, z\})$. Then by Sen’s $\alpha$, $y \in c(\{x, y\})$. By $y \not\succ x$ we have $x \in c(\{x, y\})$. By Sen’s $\beta$ this implies that $x \in c(\{x, y, z\})$.

Suppose that $z \in c(\{x, y, z\})$. Then by Sen’s $\alpha$, $z \in c(\{y, z\})$. By $z \not\succ y$ we have $y \in c(\{y, z\})$. By Sens’ $\beta$ this implies that $y \in c(\{x, y, z\})$. By the previous argument this implies that $x \in c(\{x, y, z\})$. 
We also need to show that for each $A \in P(X)$, $c(A) = c(A, \succ)$.

1. Suppose $x \in c(A)$. Then by Sen’s $\alpha$, $x \in c(\{x, y\})$ for all $y \in A$. Thus for all $y \in A$, $y \not\succ x$. So $x \in c(A, \succ)$.

2. Suppose $x \in c(A, \succ)$. Then for all $y \in A$, $y \not\succ x$. So for all $y \in A$, $x \in c(\{x, y\})$. Suppose $x \not\in c(A)$. Then there is some $z \in A$, $z \neq x$ such that $z \in c(A)$. By Sen’s $\alpha$, $z \in c(\{x, z\})$. Then $c(\{x, z\}) = \{x, z\}$, $\{x, z\} \subset A$ and $z \in c(A)$. So by Sen’s $\beta$, $x \in c(A)$. A contradiction.

So we know,

$$[\text{Sen’s } \alpha \text{ and } \beta \text{ for } c(\cdot)] \iff [c(\cdot) = c(\cdot, \succ) \text{ for the preference relation } \succ]$$
There is an alternative equivalent way to state Sen’s $\alpha$ and $\beta$.

This is Houthaker’s Axiom which is also called the *Weak Axiom of Revealed Preference (WARP).*

**WARP:** If $x$ and $y$ are both in $A$ and $B$ and if $x \in c(A)$ and $y \in c(B)$, then $x \in c(B)$ and $y \in C(A)$.

**Proposition.** $c(\cdot)$ satisfies Sen’s $\alpha$ and $\beta$ if and only if it satisfies WARP.
Partial Orders

Completeness of $\succeq$ is questionable from both a descriptive and a normative point of view.

**Definition.** $\succ$ is a partial order if it is an asymmetric and transitive binary relation.

We can define a choice function as before. What properties does it have? Sen’s $\alpha$ still holds, but Sen’s $\beta$ may fail. (On homework 1.)

Now we would not want to define $\sim$ as before. $x \not\succ y$ and $y \not\succ x$ could express indifference or non-comparability.

An alternative approach is to include a positive expression of indifference, i.e. preferences described by the pair $(\succ, \sim)$. 