Utility Functions in Economics

- Consumption set $X = \mathbf{R}^L_+$
- Prices $p \in \mathbf{R}^L_{++}$
- Wealth $w \in \mathbf{R}_{++}$
- Budget set $B(p, w) = \{x \in X : p \cdot x \leq w\}$

Consumer's Problem:

(1) x^* is **optimal** if $x^* \in B(p, w)$ and $x^* \succeq x$ for all $x \in B(p, w)$.

If u represents $\succeq$ an equivalent problem is:

(2) x^* is **optimal** if x^* solves $\max_x u(x)$ subject to $x \in B(p, w)$.

We can apply the tools of optimization theory to (2) to characterize the solution.

Useful Properties of Utility for Optimization

1. **Continuity.** The relation $\succeq$ is **continuous** if for all $x \in X$, the sets $\{y \in X : x \succeq y\}$ and $\{y \in X : y \succeq x\}$ are closed. If $\succeq$ is complete, transitive and continuous then there is a continuous utility function representing $\succeq$. Debreu [1959]

2. **Differentiability.** Smoothness assumptions on $\succeq$ are sufficient to yield existence of a differentiable utility function. Debreu [1972]

3. **Monotonicity.** The relation $\succeq$ is **strongly monotonic** if for all $x, y \in X$, $x \geq y, x \neq y$ implies $x \succ y$. If $\succeq$ is strongly monotonic then any utility function representing $\succeq$ is strictly increasing, i.e. for all $x, y \in X$, $x \geq y, x \neq y$ implies $u(x) > u(y)$.

4. **Concavity.** The function $u : \mathbf{R}^L_+ \to \mathbf{R}$ is **concave** if for all $x, y \in \mathbf{R}^L_+$ and $\alpha \in [0, 1]$, $u(\alpha x + (1 - \alpha)y) \geq \alpha u(x) + (1 - \alpha)u(y)$.

What would preferences have to look like in order to have a concave representation?

The preference relation $\succeq$ is **convex** if for all $x \in X$ the set $\{y \in X : y \succeq x\}$ is a convex set.

Suppose that $\succeq$ is convex and that u is a utility function representing $\succeq$. Then $\{x \in X : u(x) \geq k\}$ is a convex set for all k. This is weaker than concavity.

No surprise—any strictly increasing function of a utility function representing $\succeq$ still represents $\succeq$.

If $\succeq$ is convex does there exist a concave u representing $\succeq$? An additional curvature condition is needed. Fenchel [1953]

The preference relation $\succeq$ is **strictly convex** if $x \succeq y$, $x \neq y$ and $\alpha \in (0, 1)$ implies $\alpha x + (1 - \alpha)y \succ y$.

Basic Results

Suppose $X = \mathbf{R}^L_+$, $p \in \mathbf{R}^L_{++}$, $w \in \mathbf{R}_{++}$ and $\succeq$ is complete, transitive and continuous.

1. There is a solution, x^*, to the consumer's problem. [Weierstrass Theorem]

2. If in addition, $\succeq$ is strongly monotonic then $p \cdot x^* = w$.

3. If in addition, $\succeq$ is strictly convex then x^* is unique and there is a continuous function

   $$x^* : \mathbf{R}^L_{++} \times \mathbf{R}_{++} \to \mathbf{R}^L_+$$

   such that $x^*(p, w)$ solves the consumer's problem at price p and wealth w. [Uniqueness is easy. Continuity of x^* is not easy—it follows from Berge's Maximum Theorem]

   We call the function $x^*(\cdot)$ the consumer's demand function.

Characterize the Demand Function

This topic for background reading only.

Suppose that $\succeq$ is complete, transitive, strongly monotonic, strictly convex and can be represented by a twice continuously differentiable utility function u.

The consumer's problem is

$$\max u(x)$$
$$\text{s.t. } p \cdot x = w, x \geq 0$$

Intuition for 2-dimensions:

1. An indifference curve (a level set of u) is $x_2(x_1)$ such that $u(x_1, x_2(x_1)) = c$ where c is a constant. The slope of this curve is $x_2' = -u_1/u_2$, using the notation u_i to denote the partial derivative of u with respect to x_i.
2. A solution to the consumer's problem will be interior, at $x^* >> 0$, if indifference curves do not intersect the axis. $[u_i(x) \to \infty$ as $x_i \to 0$ for all i and all $x]$

3. At such an optimum, x^*, the slope of the indifference curve through x^* and the constraint $p \cdot x = w$ must be equal. So $-u_1(x^*)/u_2(x^*) = -p_1/p_2$.

4. Rewriting this we have $u_1(x^*)/p_1 = u_2(x^*)/p_2 = \lambda^*$ where λ^* is the common value of the ratio.

5. So at an interior optimum, (x^*, λ^*) solves

$$p \cdot x^* - w = 0$$
$$u_i(x^*) - \lambda^* p_i = 0 \quad \text{for all i.}$$

6. The equation system above is necessary for any L. [The Lagrange Theorem]

7. We have $L + 1$ equations in $L + 1$ unknowns. Given our assumptions there is a unique solution and it is the consumer's optimum. This defines solution functions $x^*(p, w), \lambda^*(p, w)$.

8. To describe how a change in an exogenous variable p_i or m affects the solution we can substitute the solution functions into (5) and differentiate.

A Simple Example

Suppose $u(x_1, x_2) = f_1(x_1) + f_2(x_2)$, and $f_i'(x_i) > 0$ and $f_i''(x_i) < 0$ for all x_i and all i.

[Interpretation—x_t consumption in period t.]

We can write the consumer's problem as

$$\max_{x_1} f_1(x_1) + f_2(\frac{w - p_1 x_1}{p_2})$$

At an optimal choice

$$f_1'(x_1) + f_2'(\frac{w - p_1 x_1}{p_2})(\frac{-p_1}{p_2}) = 0$$

Let $x_1^*(p, w)$ be the solution. So

$$f_1'(x_1^*(p, w)) + f_2'(\frac{w - p_1 x_1^*(p, w)}{p_2})(\frac{-p_1}{p_2}) \equiv 0$$

Differentiate with respect to w and solve to get

$$\frac{\partial x_1^*}{\partial w} = \frac{f_2'' p_1}{f_1''(p_2)^2 + f_2''(p_1)^2} > 0$$

Increasing wealth, w, increases the demand for x_1. By symmetry $\frac{\partial x_2^*}{\partial w} > 0$.

The Value of the Problem

Suppose that $\succeq$ is complete, transitive, strongly monotonic, and can be represented by a continuous utility function u.

Define the **indirect utility function**

$$V(p,m) = \max u(x)$$
$$\text{s.t. } p \cdot x = w,\ x \geq 0$$

It is easy to show that V is non-increasing in p, strictly increasing in w and that the set $\{(p,w) : V(p,w) \leq u\}$ is convex for any u.

Suppose that $V(p',w) = V(p'',w)$ and $\hat{p} = p'/2 + p''/2$.

Which environment, $\hat{p}$ or p', does the consumer prefer?

The Dual Problem

Since V is strictly increasing in w we can invert it. Let $e(p,u)$ be this inverse.

Or equivalently, we can define the **expenditure function**

$$\begin{aligned} e(p,u) = \min p \cdot x \\ \text{s.t. } u(x) \geq u,\ x \geq 0 \end{aligned}$$

It is easy to show that u is non-decreasing in p_i, strictly increasing in u and concave in p.

References

Debreu, 1959, *The Theory of Value*, New York, Wiley.

Debreu, 1972, "Smooth Preferences", *Econometrica.*

Fenchel, 1953, *Convex Cones, Sets, and Functions*, Princeton, Princeton University.