1. Using Axioms 1 through 3 and the definition, show that for every set $A$ the conditional preference order $\succeq_A$ on $L$ is Complete, transitive and reflexive.

**Answer:** Given $f$ and $g$, choose an arbitrary $h$. Since $\succeq$ is complete, either $f A h \succeq g_A h$ or $g_A h \succeq f_A h$. It follows that either $f \succeq_A g$ or $g \succeq_A f$. Thus, $\succeq_A$ is complete. Similarly, since $f_A h \succeq f_A h$, it follows that $f \succeq h$, so $\succeq_A$ is reflexive. Finally, if $f \succeq_A f'$ and $f' \succeq_A f''$, then $f_A h \succeq f_A h$ and $f_A h \succeq f_A h$. By the transitivity of $\succeq$, it follows that $f_A h \succeq f_A h$. So $f \succeq_A f''$.

2. Show that if $\succeq$ has an expected utility representation with payoff function $u$ and probability distribution $p$, and if event $A$ is not null, then $f \succeq_A g$ iff the conditional expected utility under $p$ of $f$ given $A$ exceeds that of $g$.

**Answer:** Given acts $f$ and $g$, we have $f \succeq_A g$ iff for some act $h$, $f_A h \succeq_A g_A h$. This in turn will be true iff

$$\sum_{s \in A} u(f(s))p(s) + \sum_{s \in A^c} u(h(s))p(s) > \sum_{s \in A} u(g(s))p(s) + \sum_{s \in A^c} u(h(s))p(s)$$

which is true iff $\sum_{s \in A} u(f(s))p(s) > \sum_{s \in A} u(g(s))p(s)$

Dividing each side of the last inequality by $\sum_{s \in A} p(s)$ proves the claim.

3. Show that is null, and that if $A$ is null and $B \subseteq A$, then $A$ is null. Show that if $A$ and $B$ are null, then so is $A \cup B$. (Hint: Show this first for disjoint $A$ and $B$.)

**Answer:** $f_A h = h$ for all $h$. Thus, the fact that is null now follows from the reflexivity of $\succeq$.

For the second part, first suppose that $A$ and $B$ are disjoint. $f_A h \sim g_A h$ for all $h$ so choose $h = f_B k$. Then $f_A f_B k \sim g_A f_B k$. A similar argument on $B$ shows that $f_B g_A \sim g_B g_A k$. Transitivity of $\sim$ implies $f_A \cup_B k = f_A f_B k \sim g_A g_B k = g_A \cup_B k$ for all $k$. Thus for all $f$ and $g$, $f \sim_A \cup_B g$.

Next, we show that if $B \subseteq A$ and $A$ is null, then $B$ is null. For any acts $f$ and $h$, $f_B h = f_B h_{A/B} h$. If $A$ is null, then for all acts $f$, $g$ and $h$, $f_B h \sim_A g_B h$, and so $f \sim_B g$. Thus $B$ is null.
Finally, note that if $A$ and $B$ are null, since $B - A$ is a subset of $B$, it follows that $B - A$ is null. $A$ and $B - A$ are disjoint, so $A \cup (B - A)$ is null by the first part of the proof. But $A \cup (B - A) = A \cup B$, so $A \cup B$ is null.

4. Consider a decision problem with two states and three actions: The

<table>
<thead>
<tr>
<th></th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>0</td>
<td>-10</td>
<td>-4</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>-8</td>
<td>0</td>
<td>-3</td>
</tr>
</tbody>
</table>

decision maker is an expected utility maximizer with beliefs given by a probability distribution $(p, 1 - p)$ on $\{\omega_1, \omega_2\}$. (That is, $p$ is the probability of $\omega_1$.)

(a) What will she choose at each $p$ between 0 and 1.

**Answer:** The expected utility of $d_1$ is $8p - 8$; the expected utility of $d_2$ is $-10p$; the expected utility of $d_3$ is $-3 - p$. It is easy to see that for $p$ in the interval $[0, 1/3)$, $d_2$ is the optimal act. For $p = 1/3$, $d_2$ and $d_3$ are equally good. For $p$ in the interval $(1/3, 5/9)$, $d_3$ is the optimal act. For $p = 5/9$, $d_3$ and $d_1$ are equally good. Finally, in the interval $(5/9, 1]$, $d_1$ is the optimal act.

(b) How does the utility she gets from the optimal act, the *value of the problem*, vary with $p$?

**Answer:** From the answer to part (a), it follows that in the interval $[0, 1/3]$, the optimal utility is $-10p$; in the interval $[1/3, 5/9]$, the optimal utility is $-3 - p$; and in the interval $[5/9, 1]$, the optimal utility is $8p - 8$.

(c) Suppose she does not know the true probability of $\omega_1$. She thinks it equally likely that the true probability is $1/4$ or $3/4$. What will she choose and what is the value of the problem?

**Answer:** The predicted probability of state $\omega_1$ is $1/2$, so the expected payoffs to acts $d_1$, $d_2$ and $d_3$ are $-4$, $-5$ and $-3.5$, respectively. She chooses $d_3$; the value of the problem is $-3.5$.

(d) Suppose she is told that the last time the problem was posed, the true state turned out to be $\omega_2$, but that this trial is independent
of the previous trial. What will she choose and how much utility does she receive?

**Answer:** The conditional probability that for the \( \omega \)-process, \( p(\omega_2) \) is \( 3/4 \) given that \( \omega_2 \) was observed is

\[
\]

which is \( 3/4 \). So \( p(\omega_1) = 1/4 \) with probability \( 3/4 \) and \( 3/4 \) with probability \( 1/4 \). With these posterior beliefs on the process generating the observations, the probability that the next observation will be \( \omega_1 \) is \( (3/4)(1/4) + (1/4)(3/4) = 3/8 \). By part (a), the best act in this case is \( d_3 \); it’s expected utility is \(-3.375\).

(e) Same question as before, but last time it was \( \omega_1 \) rather than \( \omega_2 \).

**Answer:** The conditional probability that for the \( \omega \)-process, \( p(\omega_1) \) is \( 3/4 \) given that \( \omega_1 \) was observed is

\[
\]

which is \( 3/4 \). The probability that \( p(\omega_1) = 1/4 \) given that \( \omega_1 \) was observed is \( 1/4 \). With these posterior beliefs on the process generating the observations, the probability that the next observation will be \( \omega_1 \) is \( (3/4)(3/4) + (1/4)(1/4) = 5/8 \). With these beliefs the expected payoffs to acts \( d_1 \), \( d_2 \) and \( d_3 \) are \(-3\), \(-6.25\) and \(-3.625\), respectively. She chooses act \( d_1 \).

(f) Suppose the decision maker knows she will be told the true \( \omega \) from last time, but she does not now know what it is. What is the value of the decision problem? How does it compare with the value of the problem when she knows she will not be told what happened last time?

**Answer:** With probability \( 1/2 \) the first observation would be \( \omega_1 \), and in this case the DM has a decision problem with value \(-3\). With probability \( 1/2 \) the first observation would be \( \omega_2 \), and in this case the DM has a decision problem with value \(-3.375\) The expected value of having the information is \(-3.1875\). If she does not receive the information, she believes both states to be equally likely. The expected payoffs to acts \( d_1 \), \( d_2 \) and \( d_3 \) are \(-4\), \(-5\) and \(-3.5\), respectively. She chooses act \( d_3 \), and the problem has value \(-3.5\). So the value of the information is \(-3.375 - (-3.5) = 0.125\).
5. Construct a Bayesian network to represent that if the weather is bad ($BW$) there is some likelihood that the airport will close ($AC$) and that planes will be delayed ($PD$). If either if the latter two events happen, it is likely that you will miss your connecting flight ($MF$).

You are given the following conditional probabilities:

$\Pr(BW) = .3$
$\Pr(AC | BW) = .8$
$\Pr(AC | \neg BW) = .01$
$\Pr(PD | BW) = .7$
$\Pr(PD | \neg BW) = .1$
$\Pr(MF | AC) = 1 \Pr(MF | \neg AC \cap PD) = .6$
$\Pr(MF | \neg AC \cap \neg PD) = .1$

(a) What is the probability that a plane is delayed?
(b) What is the probability that the weather is bad, given that a plane is delayed?
(c) What is the probability that the airport is closed, given that a plane is delayed?
(d) What is the probability that you miss your flight, given that a plane is delayed?

Answer: The Bayesian network has this form:

```
    BW
   /
  / \
AC  PD
 \ /
   MF
```

(a) $\Pr(PD) = \Pr(PD | BW) \Pr(BW) + \Pr(PD | \neg BW) \Pr(\neg BW) = .7 \times .3 + .1 \times .7 = .28.$
(b) $\Pr(BW | PD) = \frac{\Pr(PD | BW) \Pr(BW)}{\Pr(PD)} = .7 \times .3 / .28 = 3/4.$
The fact that $\Pr(AC \mid BW \cup PD) = \Pr(AC \mid BW)$ follows from the Bayesian network, since $AC$ is conditionally independent of $PD$ given $BW$; similarly with $BW$ replaced by $\neg BW$.

(d)

$$
\Pr(MF \mid PD) = \Pr(MF \mid PD \cup AC) \Pr(AC \mid PD) + \Pr(MF \mid PD \cup \neg AC) \Pr(\neg AC \mid PD)
$$

$$
= .6025 + .6 \times .3975 = .841.
$$

6. Show that there is no additive representation for the health-wealth example in Prof. Halpern’s lecture notes on graphical representations of utility.

**Answer:** Suppose there were an additive representation and $f_1(h) = a_1, f_1(\overline{h}) = a_2, f_2(w) = b_1$, and $f_2(\overline{w}) = b_2$. Then we must have $a_1 + b_1 = 5, a_1 + b_2 = 2, a_2 + b_1 = 1, \text{ and } a_2 + b_2 = 0$. Simple algebra shows that we must have $b_2 = -a_2, b_1 = 1 - a_2, \text{ and } a_1 = 2 + a_2$. It follows that $a_1 + b_1 = 3 \neq 5$, so these equations have no solution.