CS**5643**

13 Brief survey of fluid animation

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Fluid simulation landscape

Eulerian methods

- fluid properties are written down at points that stay fixed while the fluid moves by
- more classical approach, easier to make accurate in the limit, hard to make fast

Lagrangian methods

- fluid properties are written down at points that move with the fluid
- can be simpler, easier to make fast, harder to achieve high accuracy

Hybrid methods

- convert back and forth between Eulerian and Lagrangian representations in hopes of getting the advantages of both

Leonhard Euler
(1707–1783)

Joseph-Louis Lagrange
(1736–1813)

Outline

Elements of fluid mechanics

- reference: Bridson & Müller-Fischer SIGGRAPH course notes

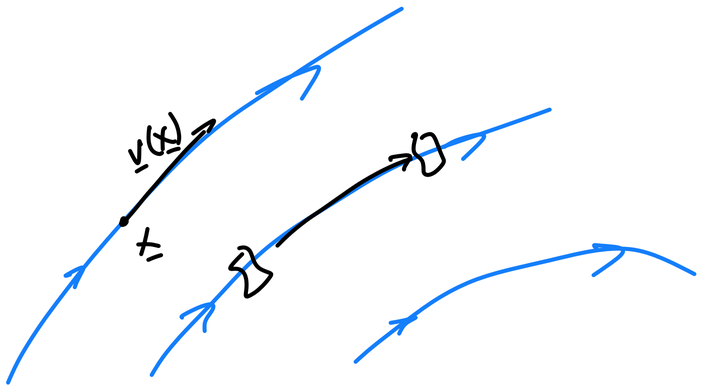
Smoothed Particle Hydrodynamics

- the basic scheme for particle-based (Lagrangian) fluids
- particles represent bits of fluid mass
- interact via inter-particle forces

Grid based fluids

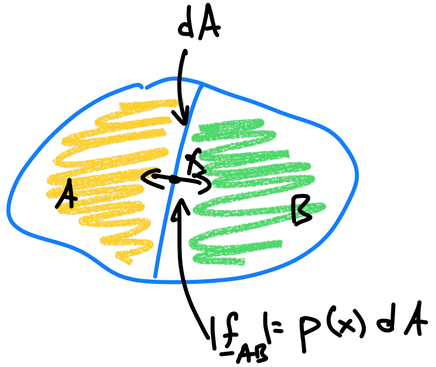
- represent fluid velocity and pressure on grids
- each timestep updates velocity and pressure fields
- operate in alternating advection (move along velocity) and projection (solve for pressure) steps

Key ideas of fluid mechanics



Velocity $\mathbf{u}(\mathbf{x})$

- blob at position $\mathbf{x}$ will be at position $\mathbf{x} + \mathbf{u}(\mathbf{x})dt$ at time dt



Pressure $p(\mathbf{x})$

- blob at position $\mathbf{x}$ exerts force $p(\mathbf{x})\,dA$ on a neighboring blob

Incompressibility

- water under most conditions and air under mild conditions do not appreciably compress
- thus pressure instantly adjusts to keep the volume density constant

Viscosity

- water an air have very little viscosity; honey or motor oil have higher viscosity

Material derivative

A quantity q in a fluid can change for two reasons

- because different bits of fluid show up where we are looking
- because the actual properties of the bits of fluid are changing

Measuring what is changing about the fluid itself

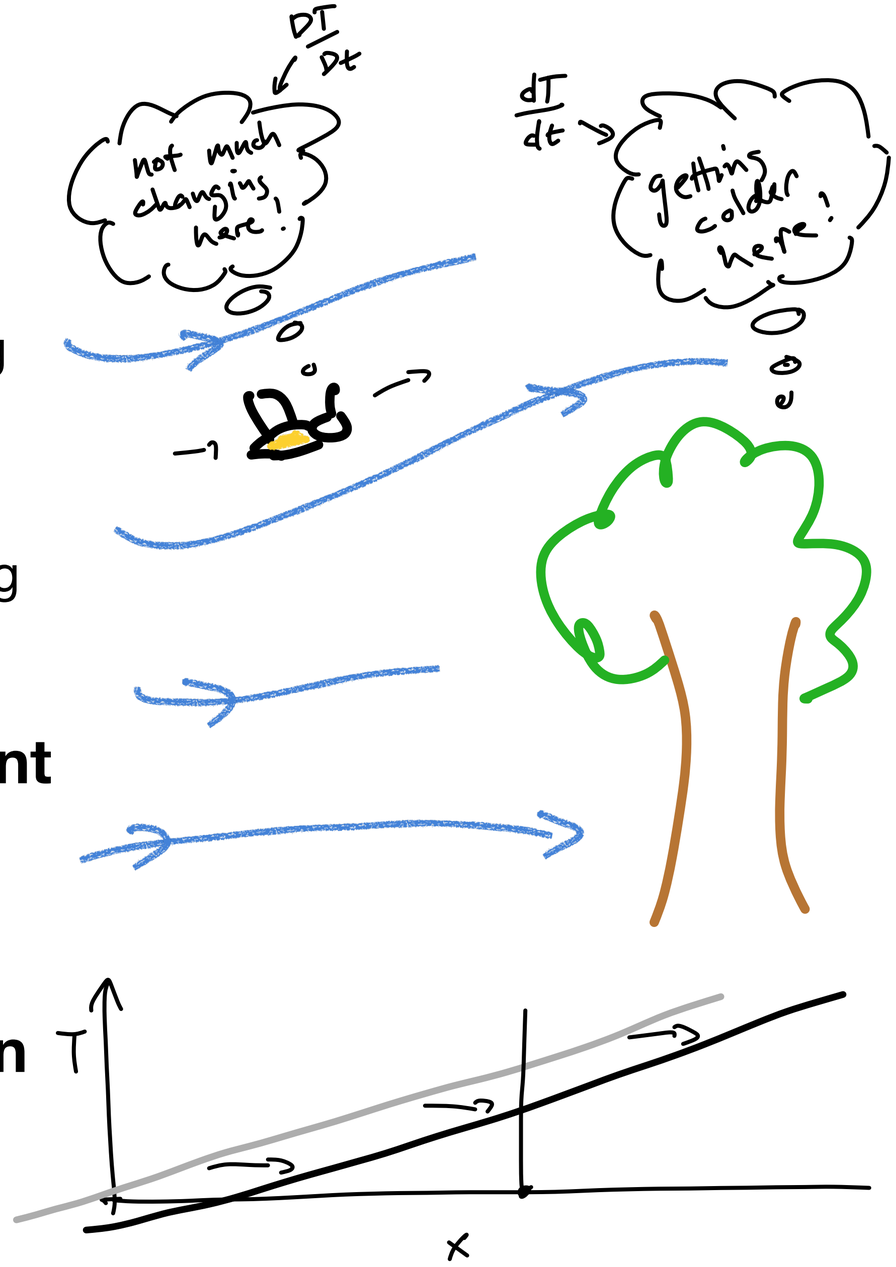
- *material derivative* Dq/Dt is the time derivative seen by a bug riding with the fluid

Measuring what changes at a fixed observation point

- ordinary time derivative dq/dt is the derivative seen by an observer at a fixed position

E.g.: temperature gradient in 1D, steady fluid motion

- $DT/Dt = 0$ but $dT/dt = -(dT/dx)u(x)$



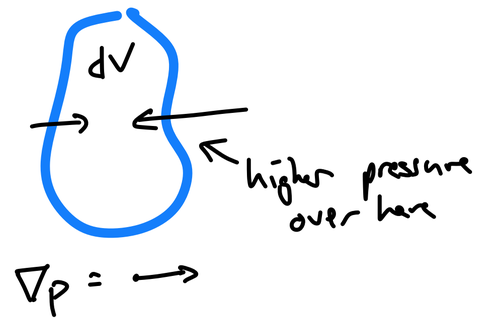
Rules for fluid motion

…for incompressible fluids with negligible viscosity

First rule: $\mathbf{f} = m\mathbf{a}$

const.

- consider a little blob of fluid with volume dV. It has mass $m = \rho dV$.
- it experiences a net force from its neighbors of $-\nabla p(\mathbf{x})\, dV$.
- it experiences a body force, e.g. due to gravity, of $m\mathbf{g}$.
- the acceleration of this blob of fluid is governed by
  - $m\dfrac{D\mathbf{u}(\mathbf{x})}{Dt} + \nabla p(\mathbf{x}) dV = m\mathbf{g}$ or $\rho\dfrac{D\mathbf{u}(\mathbf{x})}{Dt} dV + \nabla p(\mathbf{x})\, dV = \rho\mathbf{g}\, dV$
- in the limit as $dV \to 0$ we have
  - $\dfrac{D\mathbf{u}(\mathbf{x})}{Dt} + \dfrac{1}{\rho}\nabla p(\mathbf{x}) = \mathbf{g}$ which is the momentum balance equation

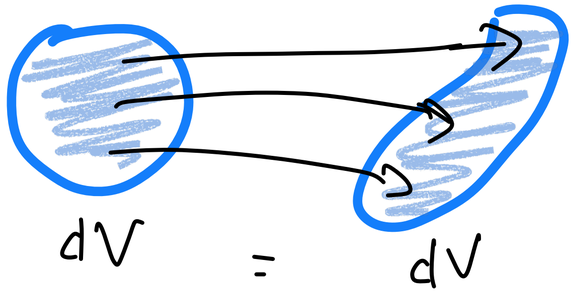


Rules for fluid motion

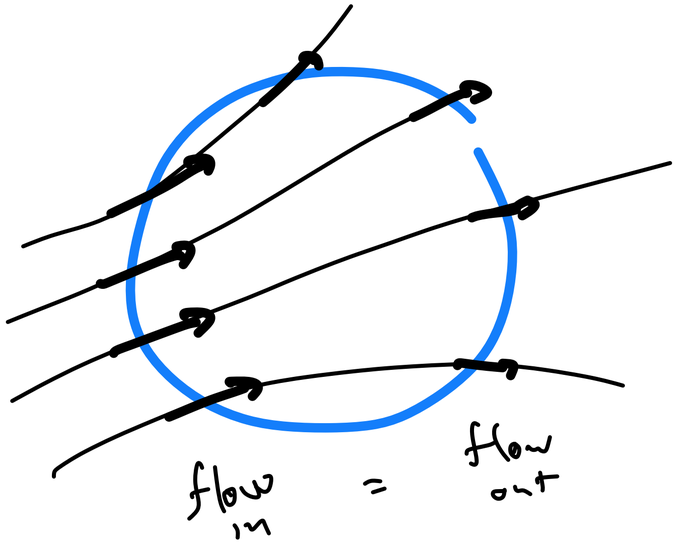
Second rule: conservation of mass

- since the density is constant, the flows into and out of a region must balance
  - $\int_{\partial\Omega} \mathbf{u}(\mathbf{x}) \cdot d\mathbf{n} = 0 = \int_{\Omega} \nabla \cdot \mathbf{u}(\mathbf{x})\, d\mathbf{x}$ —by the divergence theorem
- this has to be true for all volumes so the velocity field is divergence free
  - $\nabla \cdot \mathbf{u}(\mathbf{x}) = 0$

Lagrange:



Euler:



Rules for fluid motion

So the equations that define the problem are

$$\frac{D\mathbf{u}(\mathbf{x})}{Dt} + \frac{1}{\rho} p(\mathbf{x}) = \mathbf{g}$$

$$\nabla \cdot \mathbf{u}(\mathbf{x}) = 0$$

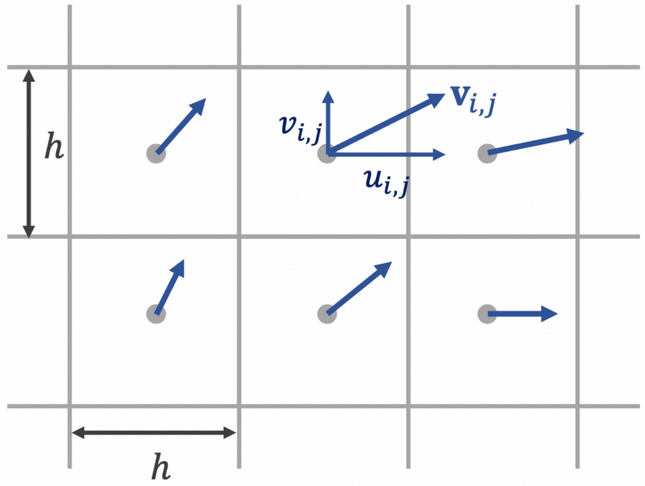
- these are the Euler equations for an incompressible and inviscid fluid
  (add viscosity and they are called Navier-Stokes equations)

Discretizing the Euler equations

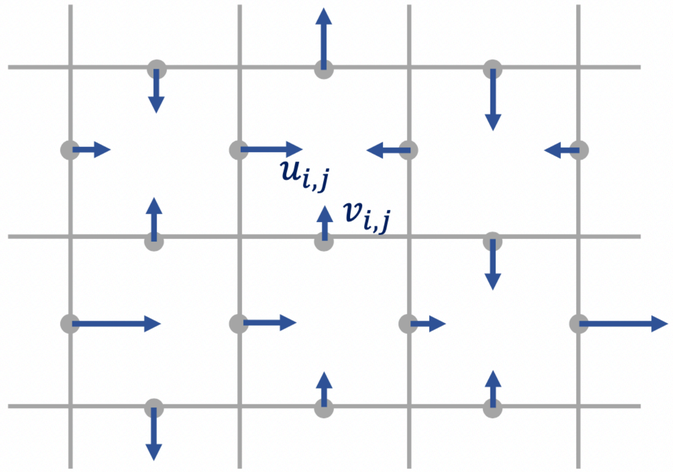
In some ways the grid-based Eulerian approach is the easiest to write down

We need to store velocity $\mathbf{u}(\mathbf{x})$ and store (or at least think about) pressure $p(\mathbf{x})$

- we will need to relate derivatives of pressure to velocity and vice versa
- it is convenient then to use a staggered rectangular grid
  - associate p with the cells of a regular grid
  - associate components of $\mathbf{u}$ with the edges of the grid



collocated grid



staggered grid

Matthias Müller

Solving the Euler equations

Successful algorithms treat terms of Euler equations separately

- first solve $\frac{D\mathbf{u}}{Dt} = 0$ to get $\mathbf{u}_A$ from $\mathbf{u}_k$

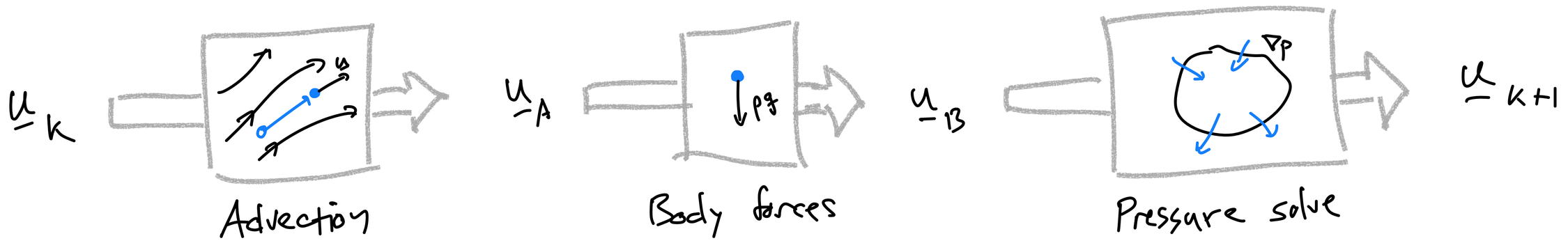
  (advection step: move velocity along the velocity)

- next add $\mathbf{g}$ to get $\mathbf{u}_B$ from $\mathbf{u}_A$

  (velocity update: can also add other body forces if desired)

- finally solve for pressure and update $\frac{d\mathbf{u}}{dt} = \frac{1}{\rho}\nabla p$ to get $\mathbf{u}_{k+1}$

  (projection: ensure $\mathbf{u}$ is divergence free)



Grid fluid solver: advection step

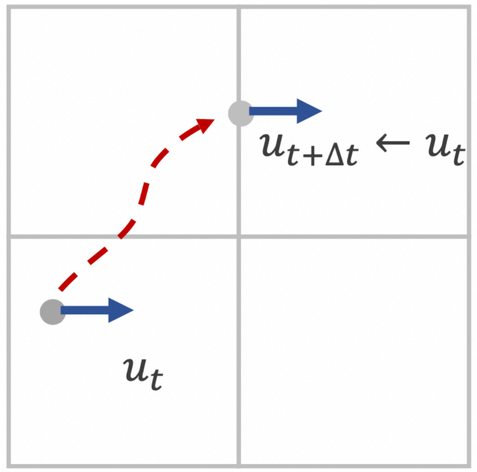
For the advection step we want $\mathbf{u}(\mathbf{x})$ to be the velocity that the bit of fluid at $\mathbf{x}$ had a time h ago

- one approach: work out in terms of space and time derivatives of $\mathbf{u}$ on the grid
  - sadly this requires very small time steps for stability
- better approach: simply look where this bit of fluid was a time h ago
  - step backward by h along the velocity vector
  - sample (interpolate) the fluid velocity at that point to get $\mathbf{u}_A$

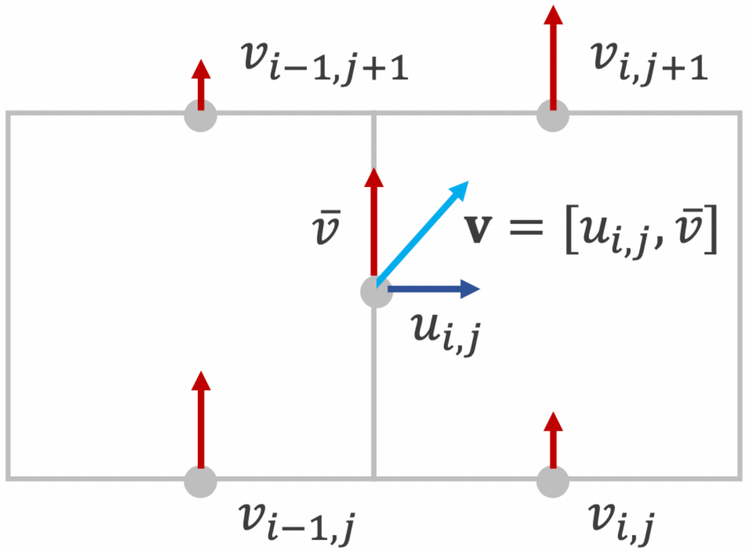
This approach is known as “semi-Lagrangian advection”

- it is unconditionally stable since it’s strictly interpolating; range of velocities gets smaller with each step, never larger
- the same approach can be used for other quantities as well as the velocity itself

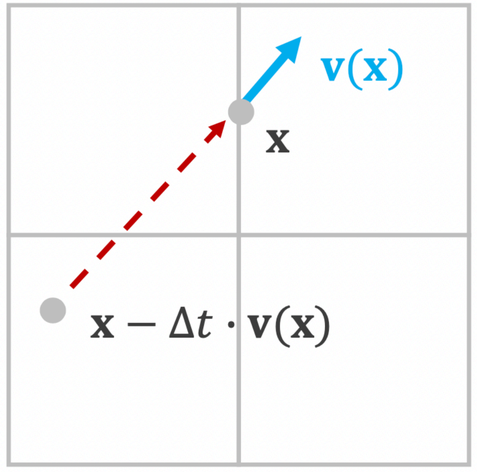
Semi-Lagrangian advection



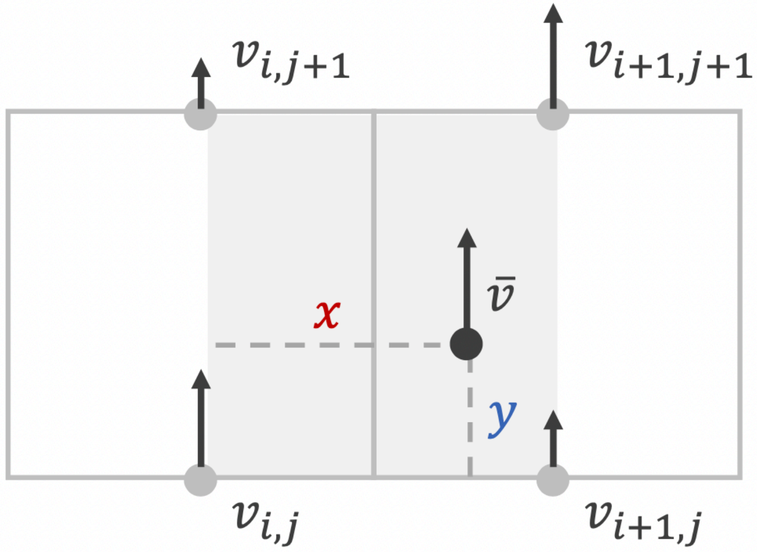
Trace back in time to a (non-grid) sample location at the previous time step



To get the velocity vector you need to interpolate the components from the staggered grid



Compute sample location using a simple Forward Euler step



To sample the previous velocity you need to linearly interpolate from the corresponding velocity grid

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Velocity update

After the advection step we have new velocities everywhere

- they correspond to each bit of fluid flying straight
- no interaction between bits of fluid
- no concern for forces

First and simplest fix: add acceleration due to body forces

- $\mathbf{u}_B(\mathbf{x}) = \mathbf{u}_A(\mathbf{x}) + h\mathbf{g}$ — in practice for gravity only have to update the y component

Velocity projection (pressure update)

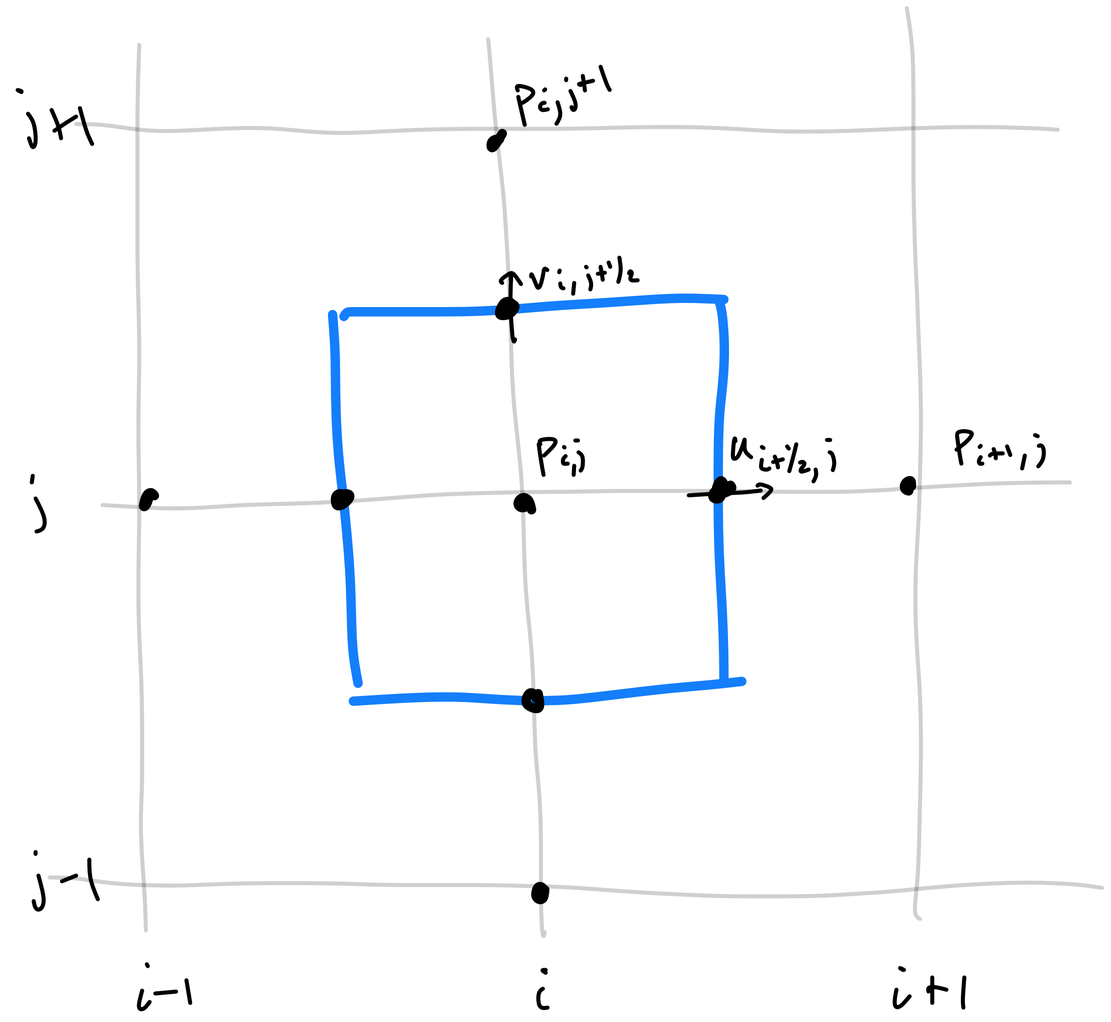
After advection and velocity update, velocity field is broken

- it is supposed to be divergence free (since the fluid is incompressible)
- in an incompressible fluid the pressure acts to enforce this constraint

Last substep is to solve for the pressure that will cause $\nabla \cdot \mathbf{u} = 0$

- we want a pressure field so that $\nabla \cdot \mathbf{u}_{k+1} = \nabla \cdot \left(\mathbf{u}_B(\mathbf{x}) - \frac{h}{\rho} \nabla p(\mathbf{x}) \right) = 0$
- this expands to $h\Delta p = -\rho \nabla \cdot \mathbf{u}_B$ where $\Delta p = \nabla \cdot \nabla p$ is the Laplacian of p
- this is a Poisson problem and there are good methods to solve it
- result is the final $\mathbf{u}_{k+1}$ for the next timestep
  (pressure does not need to be saved)

Velocity projection (pressure update)



update to velocity will be:

$$u^{+}_{i+1/2,j} = u^{-}_{i+1/2,j} - \frac{\Delta t}{\rho}(p_{i+1,j} - p_{i,j})/\Delta x$$

divergence apx is:

$$\nabla \cdot u_{i,j} \simeq \left(\frac{u_{i+1/2,j} - u_{i-1/2,j}}{\Delta x}\right) + \left(\frac{v_{i,j+1/2} - v_{i,j-1/2}}{\Delta x}\right) = 0$$

pressure must satisfy:

$$\frac{\Delta t}{\rho}\left[\frac{4p_{i,j} - p_{i+1,j} - p_{i,j-1} - p_{i+1,j} - p_{i,j+1}}{\Delta x^2}\right] =$$

$$-\nabla \cdot u^{-}_{i,j}$$

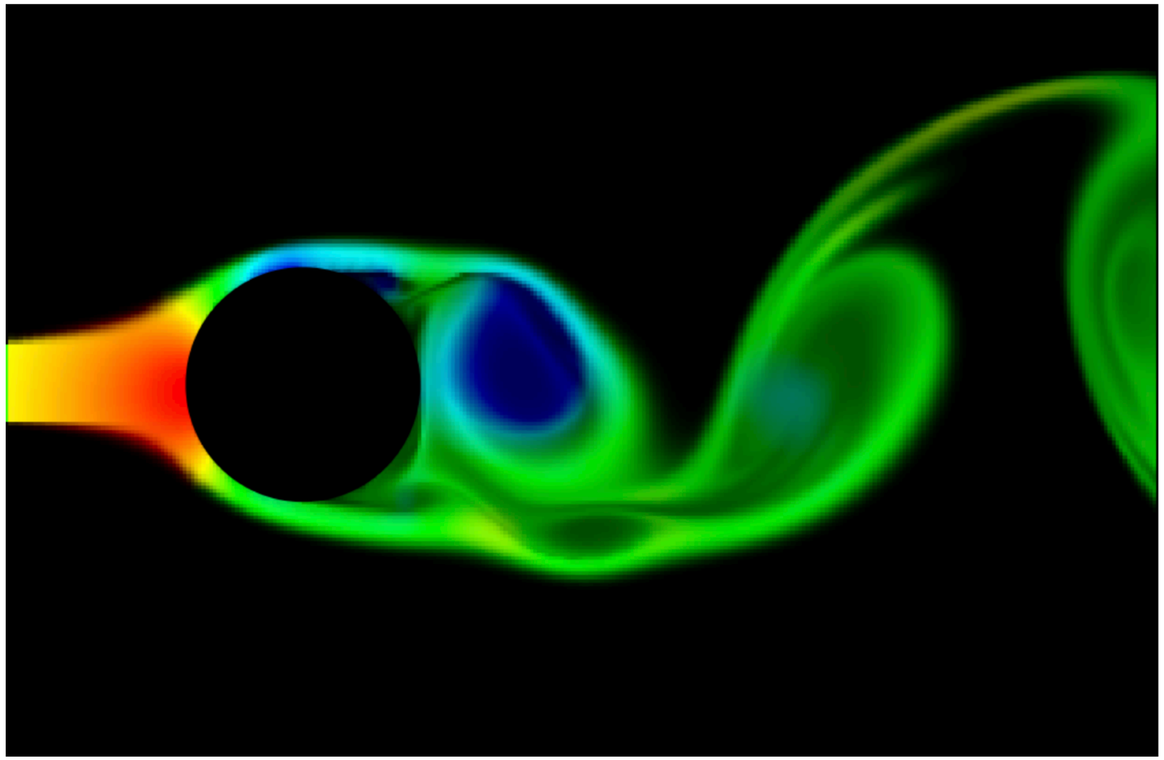
Grid simulator

Will need boundary conditions

- pressure = 0 (Neumann) conditions for free boundaries (open space)
- normal velocity = 0 (Dirichlet) conditions for solid boundaries (obstacles)
- require simple local changes in the pressure step

Those are all the pieces required

- often you'll want to advect smoke density
  or temperature passively along with the fluid
- simple approximation: upward buoyancy force
  proportional to temperature
- demo by Matthias Müller



Particle based fluids

A whole different approach to fluid simulation

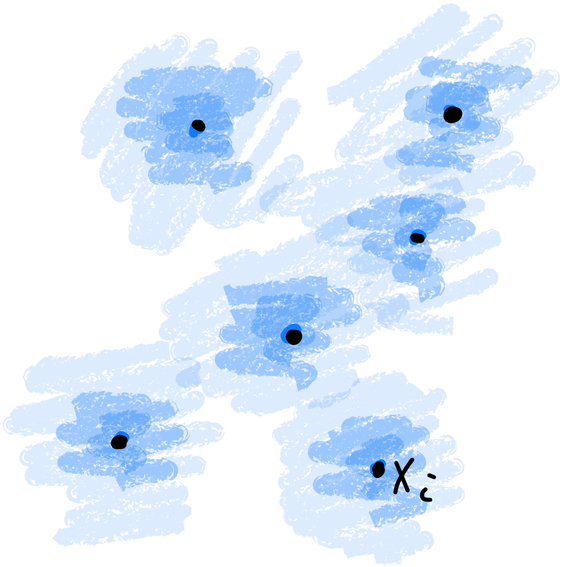
- represent fluid variables on movable particles
- advect the particles with the velocity (ballistic motion)
- use inter-particle forces to manage pressure and viscosity

Key mathematical formulation: Smoothed Particle Hydrodynamics (SPH)

- methods originated in astrophysics in the 90s
- idea: define continuously varying quantities by blurring the particles
- sample these quantities at particle locations to compute forces

Defining a continuous field through smoothing

Use a smooth kernel



- popular choice: $W_{\text{poly6}}(r) = \frac{315}{64\pi d^9} \begin{cases} (d^2 - r^2)^3 & 0 \le r \le d \\ 0 & \text{otherwise,} \end{cases}$
- define kernel by radial falloff: $W(\mathbf{r}) = W(\|\mathbf{r}\|)$
- example: fluid density
  - $\rho(\mathbf{x}) = \sum_i m_i W(\mathbf{x} - \mathbf{x}_i)$ — and let $\rho_i = \rho(\mathbf{x}_i)$ be the density at particle i
- once we have density, define interpolants of any quantity by generalizing this as
  - $q(\mathbf{x}) = \sum_i \frac{m_i}{\rho_i} q_i W(\mathbf{x} - \mathbf{x}_i)$
- this provides machinery to define the terms in the Navier-Stokes equations

Computing forces in SPH

Divergence-free velocity is not exactly maintained

- this means we can't assume no density variations

…but mass is exactly conserved

- they are particles, they can't help it
  (so no need to worry about losing mass even with density fluctuations)

Particles behave like regular ballistic particles subject to forces

- $$\rho \frac{D\mathbf{u}(\mathbf{x})}{Dt} = -\nabla p(\mathbf{x}) + \rho \mathbf{g}$$
- the left hand side defines the particle acceleration
- SPH lets us compute the forces on the right hand side
- gravity (or other external forces) is easy

Computing the pressure force

Pressure exerts a force on each particle

- the pressure is a function of the nearby particles
- given pressure p_i for each particle the interpolant is
  - $$p(\mathbf{x}) = \sum_i \frac{m_i}{\rho_i} p_i W(\mathbf{x} - \mathbf{x}_i)$$
- and its gradient is
  - $$\nabla p(\mathbf{x}) = \sum_i \frac{m_i}{\rho_i} p_i \nabla W(\mathbf{x} - \mathbf{x}_i)$$
- resulting in the force due to pressure on particle i
  - $$\mathbf{f}_i^{\mathrm{p}} = -\nabla p(\mathbf{x}_i) = -\sum_j \frac{m_j}{\rho_j} p_j \nabla W(\mathbf{x}_i - \mathbf{x}_j)$$

Computing the pressure force

Symmetrized force is typically used

- $$\mathbf{f}_i^{\mathrm{p}} = -\nabla p(\mathbf{x}_i) = -\sum_j \frac{m_j}{\rho_j}\frac{p_i + p_j}{2}\nabla W(\mathbf{x}_i - \mathbf{x}_j)$$

How to define pressures for the particles?

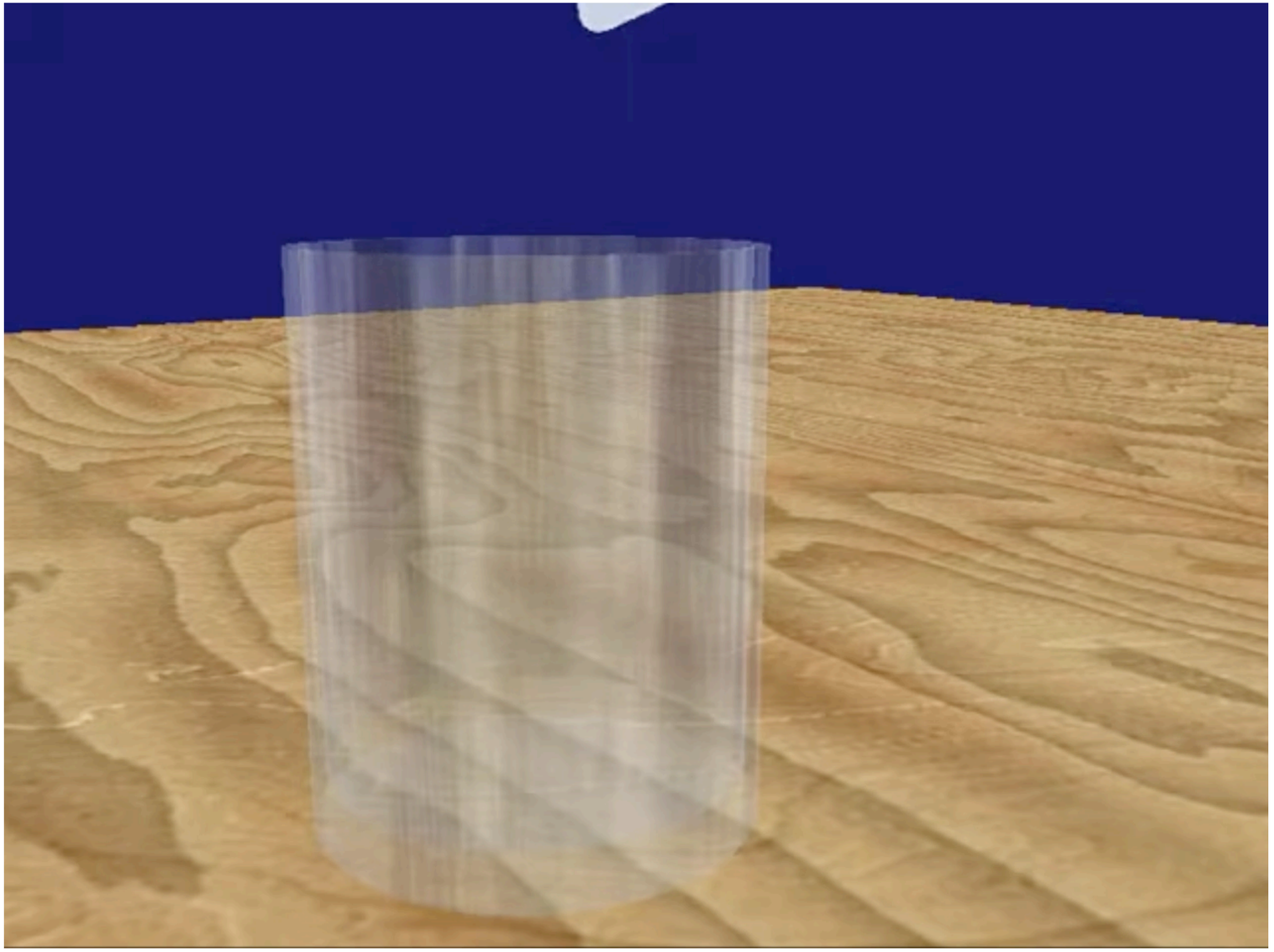
- simple: think of fluid as compressible and define pressure from density
- popular model $p(\mathbf{x}) = k(\rho(\mathbf{x}) - \rho_0)$ —kind of like ideal gas law
- advantage: simplicity
- disadvantage: springiness (if k is low) or numerical stiffness (if k is high)
- more sophisticated method: use iterative solve to find incompressible velocities

Algorithm 1 SPH / WCSPH

while animating **do**
  for all i **do**
    find neighborhoods $N_i(t)$
  for all i **do**
    compute density $\rho_i(t)$
    compute pressure $p_i(t)$
  for all i **do**
    compute forces $\mathbf{F}^{p,v,g,ext}(t)$
  for all i **do**
    compute new velocity $\mathbf{v}_i(t+1)$
    compute new position $\mathbf{x}_i(t+1)$

[Solenthaler and Pajarola 2009]

Early graphics SPH: real time c. 2003



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Iteratively solving for pressure: predictive-corrective incompressible SPH

- iterate within each timestep to find pressures that lead to velocities that lead to uniform density
- allows larger timesteps
- ultimately faster than the alternative of using tiny simtesteps to achieve sufficient pressure uniformity

Algorithm 2 PCISPH

while animating **do**
  for all i **do**
    find neighborhoods $N_i(t)$
  for all i **do**
    compute forces $\mathbf{F}^{v,g,ext}(t)$
    initialize pressure $p(t) = 0.0$
    initialize pressure force $\mathbf{F}^p(t) = 0.0$
  while $(\rho^*_{err}(t+1) > \eta)$ || $(iter < minIterations)$ **do**
    for all i **do**
      predict velocity $\mathbf{v}^*_i(t+1)$
      predict position $\mathbf{x}^*_i(t+1)$
    for all i **do**
      predict density $\rho^*_i(t+1)$
      predict density variation $\rho^*_{err}(t+1)$
      update pressure $p_i(t) \mathrel{+}= f(\rho^*_{err}(t+1))$
    for all i **do**
      compute pressure force $\mathbf{F}^p(t)$
  for all i **do**
    compute new velocity $\mathbf{v}_i(t+1)$
    compute new position $\mathbf{x}_i(t+1)$

[Solenthaler and Pajarola 2009]

SPH with better pressure solve

Barbara Solenthaler