11 Rigid body motion
Overview

Kinematics of rigid bodies (emphasis on 2D case)
- state includes position and rotation for each body

Dynamics of a free body
- how to compute time derivative of state
- forces, torques, impulses

Rigid body collisions
- isolated body-obstacle collision
- isolated body-body collision

Systems of collisions
- linear complimentarity problems
- iterative solution methods
Rigid body state

A position

• I’ll call it $\mathbf{p}$
• it’s the position of the center of mass (keeps things simpler)

A rotation

• can be represented with a rotation matrix $\mathbf{R}$
• defines the mapping from the body’s local space to world space:
  - $\mathbf{x} = \mathbf{p} + \mathbf{RX}$
Representing rigid body state

In 2D

- \( \mathbf{p} \) is simple (2 numbers)
- \( \mathbf{R} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \) so let’s write down \( \mathbf{q} = \begin{bmatrix} c \\ s \end{bmatrix} \)
- so state has 3 numbers but 2 DoF since \( \|\mathbf{q}\| = 1 \)

In 3D

- \( \mathbf{p} \) is simple (3 numbers)
- rotation is best represented as a unit quaternion
  - \( \mathbf{q} = [w \ x \ y \ z]^T \)
  - \( \mathbf{R}(t) = \mathbf{R}(\mathbf{q}(t)) \)
- so state has 7 DoF but 6 DoF since \( \|\mathbf{q}\| = 1 \)
Rigid body velocity

Motion of a point on a moving body

- \( \mathbf{x}(t) = \mathbf{p} + \mathbf{R}(t)\mathbf{X} \) (\( \mathbf{X} \) is not changing)
- \( \dot{\mathbf{x}}(t) = \dot{\mathbf{p}}(t) + \dot{\mathbf{R}}(t)\mathbf{X} = \mathbf{v}(t) + \dot{\mathbf{R}}(t)\mathbf{X} \)
- \( \dot{\mathbf{R}} \) maps a body-space point to the rotational part of its world-space velocity

Computing the derivative

- derivatives of rotations are special: \( \dot{\mathbf{R}}\mathbf{R}^T = 1 \) so
  \[
  \frac{d}{dt}\mathbf{R}\mathbf{R}^T = 0 = \dot{\mathbf{R}}\mathbf{R}^T + \mathbf{R}\dot{\mathbf{R}}^T = \dot{\mathbf{R}}\mathbf{R}^T + (\dot{\mathbf{R}}\mathbf{R}^T)^T
  \]
- so \( \dot{\mathbf{R}}\mathbf{R}^T \) is antisymmetric \[
  \begin{bmatrix}
  0 & -\omega \\
  \omega & 0
  \end{bmatrix}
  \]; call this \( \omega^\times \)
  and \( \omega \) is known as angular velocity
- \( \dot{\mathbf{R}} = \omega^\times \mathbf{R} \), and in 2D, \( \dot{\mathbf{q}} = \omega^\times \mathbf{q} \)
Forces and torques

When a force is applied to a point \( \mathbf{x} \) on a body

- the force affects the center-of-mass velocity
  - \( \mathbf{f} = m \dot{\mathbf{v}} \)

- the force also affects the angular velocity
  - effect depends on offset \( \mathbf{r} = \mathbf{x} - \mathbf{p} \)
  - only the component perpendicular to \( \mathbf{r} \) affects the body’s rotation
  - effect is proportional to \( ||\mathbf{r}|| \)
  - hence define torque \( \tau = \mathbf{r} \times \mathbf{f} \)
  - \( \tau = I \dot{\omega} \) where \( I \) is the moment of inertia
Impulses

Just like with particles, impulses cause instantaneous change in velocity

- for linear velocity, \( m\Delta v = J \) just like with a particle
- and for angular velocity, \( I\Delta \omega = r \times J \) (a torque impulse)

This will be useful for collisions

- \( v^+ = v^- + m^{-1}J \)
- \( \omega^+ = \omega^- + I^{-1}r \times J \)
Collisions: rigid body–obstacle

Body collides with fixed obstacle

- want to apply an impulse at the point of contact so that $v_n^+ = -c_r v_n^-$
- before collision: $v_n^- = \hat{n} \cdot (v^- + \omega^- \times r)$
- impulse is along normal: $J = \gamma \hat{n}$
- after collision: $v^+ = v^- + m^{-1} J$ ; $\omega^+ = \omega^- + I^{-1} r \times J$
- relate normal velocities before and after to find $\gamma$:

$$v_n^+ = \hat{n} \cdot (v^- + m^{-1} J + (\omega^- + I^{-1} r \times J) \times r)$$

$$= \hat{n} \cdot (v^- + m^{-1} \gamma \hat{n} + \omega^- \times r + I^{-1} \gamma (r \times \hat{n}) \times r)$$

$$= v_n^- + \gamma \left( m^{-1} + \hat{n} \cdot I^{-1} (r \times \hat{n}) \times r \right)$$

- so $\gamma = - (1 + c_r) m_{\text{eff}} v_n^-$ where $m_{\text{eff}} = \left( m^{-1} + \hat{n} \cdot I^{-1} (r \times \hat{n}) \times r \right)^{-1}$