CS**5643**

07 Deformation models

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Spring 2023

Working out our spring force from the energy

Start with the spring energy

- $E_{ij}(\mathbf{x}) = \frac{1}{2}k_s(\|\mathbf{x}_i - \mathbf{x}_j\| - l_0)^2$ (this is the contribution of one spring to the total system energy)

Force is minus the gradient of energy

- $\mathbf{f}_i(\mathbf{x}) = -\dfrac{\partial E}{\partial \mathbf{x}_i}(\mathbf{x})$ (remember $\mathbf{x}$ is a big vector of all the positions; this partial derivative is zero for all the particles that are not connected to this particular spring)

Take the computation one step at a time:

- derivative of $\mathbf{x}_i - \mathbf{x}_j$ is I wrt. $\mathbf{x}_i$ and $-I$ wrt. $\mathbf{x}_j$
- derivative of $\|\mathbf{v}\|$ wrt. $\mathbf{v}$ is $\hat{\mathbf{v}}$

where $\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j$

- derivative of E_{ij} wrt $\|\mathbf{v}\|$ is $k_s(\|\mathbf{v}\| - l_0)$
- put it all together: $\mathbf{f}_i = -\partial E/\partial \mathbf{x}_i = -k_s(\|\mathbf{x}_{ij}\| - l_0)\hat{\mathbf{x}}_{ij}$ and $\mathbf{f}_j = -\partial E/\partial \mathbf{x}_j = k_s(\|\mathbf{x}_{ij}\| - l_0)\hat{\mathbf{x}}_{ij}$

Alternative “variational” notation

Derivative is a linear transformation; write down the output

- instead of $\dfrac{\partial \mathbf{f}}{\partial \mathbf{x}} = A$ write $\delta \mathbf{f} = A\,\delta \mathbf{x}$
- when the matrix A is awkward to write down this can be neater…
- $\delta \mathbf{x}_{ij} = \delta \mathbf{x}_i - \delta \mathbf{x}_j$
- $\delta \|\mathbf{v}\| = \hat{\mathbf{v}} \cdot \delta \mathbf{v}$
- $\delta E = k_s(l - l_0)\delta l$
- substitute to get $\delta E = \underbrace{k_s(\|\mathbf{x}_{ij}\| - l_0)\hat{\mathbf{x}}_{ij}}_{f_i} \cdot \delta \mathbf{x}_i \underbrace{- k_s(\|\mathbf{x}_{ij}\| - l_0)\hat{\mathbf{x}}_{ij}}_{f_j} \cdot \delta \mathbf{x}_j$
- read off $\mathbf{f}_i$ and $\mathbf{f}_j$

Deformable models

Mass-spring models can get you somewhere

- but only so far
- they were used a lot back in the Old Days

They have their limitations

- hard to separate different stiffnesses (e.g. bend/shear springs contribute to stretch)
- hard to control preservation of volume in deformations
- hard to make them agree with measurements

Let’s keep the idea of deriving forces from energies

- define energies to get the behavior we want
- borrow energies from other fields to get more accurate models

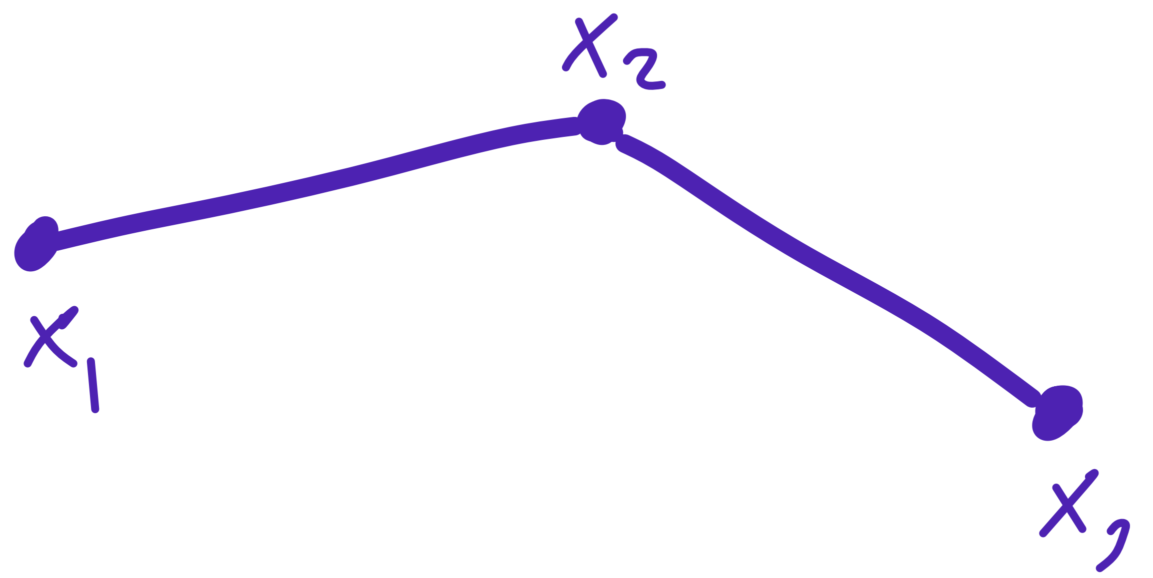
Example: hinge energy

We made a rope before using linear springs

- connect springs between every other point
- when rope bends, the springs fight one another, indirectly cause bending resistance

More direct approach

- just make the energy depend on the bending angle θ (well, $\sin\frac{\theta}{2}$)



$E = k \sin^2 \theta/2 = \frac{k}{2}(1-\cos\theta)$ equiv. $E = -\frac{k}{2}\cos\theta$.

$\cos\theta = \hat{x}_{12} \cdot \hat{x}_{23}$

$\delta \cos\theta =$

$\frac{1}{\|x_{12}\|}\left(\hat{x}_{23} - (\hat{x}_{23}\cdot\hat{x}_{12})\hat{x}_{12}\right)\cdot \delta x_{12} +$

$\frac{1}{\|x_{23}\|}\left(\hat{x}_{12} - (\hat{x}_{12}\cdot\hat{x}_{23})\hat{x}_{23}\right)\cdot \delta x_{23}$

subroutine:

$\delta(\hat{a}\cdot\hat{b}) = \hat{b}\cdot\delta\hat{a} + \hat{a}\cdot\delta\hat{b}$

$= \frac{\hat{b}}{\|a\|}\cdot(\delta a - \hat{a}(\hat{a}\cdot\delta a)) + \dots$

$= \frac{1}{\|a\|}(\hat{b}\cdot\delta a - \hat{b}\cdot\hat{a}(\hat{a}\cdot\delta a)) + \dots$

$= \frac{1}{\|a\|}(\hat{b} - (\hat{b}\cdot\hat{a})\hat{a})\cdot\delta a + \frac{1}{\|b\|}(\hat{a} - (\hat{a}\cdot\hat{b})\hat{b})\cdot\delta b$

subroutine:

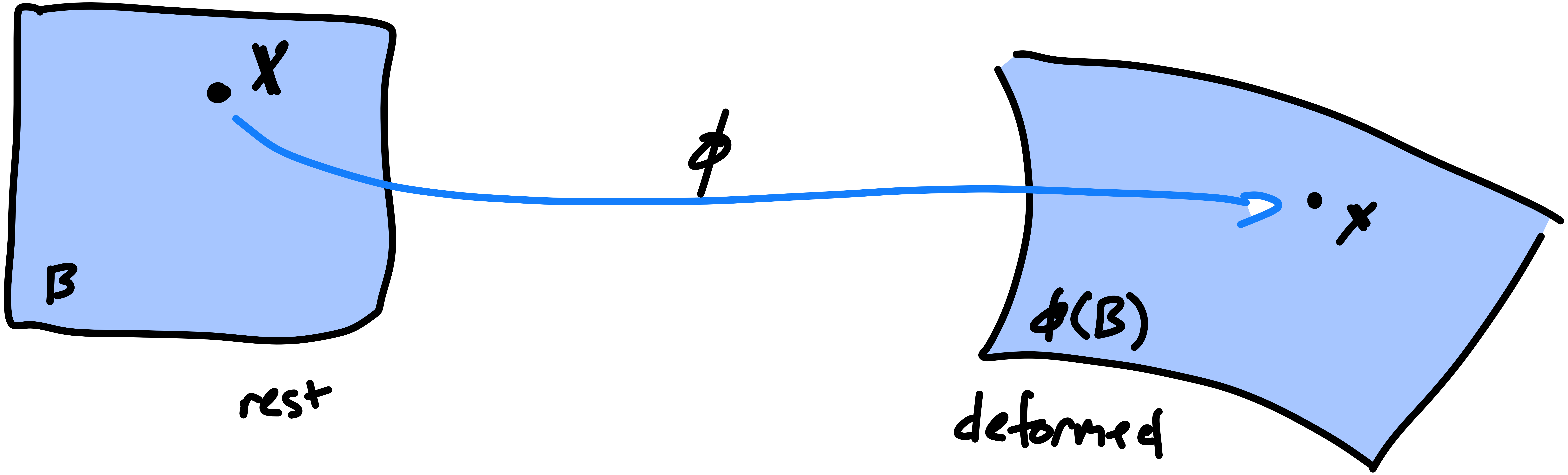
$\delta\hat{x} = \frac{1}{\|x\|}(I - \hat{x}\hat{x}^T)\delta x$

$= \frac{1}{\|x\|}(\delta x - \hat{x}(\hat{x}\cdot\delta x))$

Deformation map

A deforming object is described by a time varying function

- $\mathbf{x} = \phi(\mathbf{X}, t)$
- maps the *rest position* of a chunk of material to its current *deformed* position
- aka. a map from *material* space to *world* space
- varies as a function of time



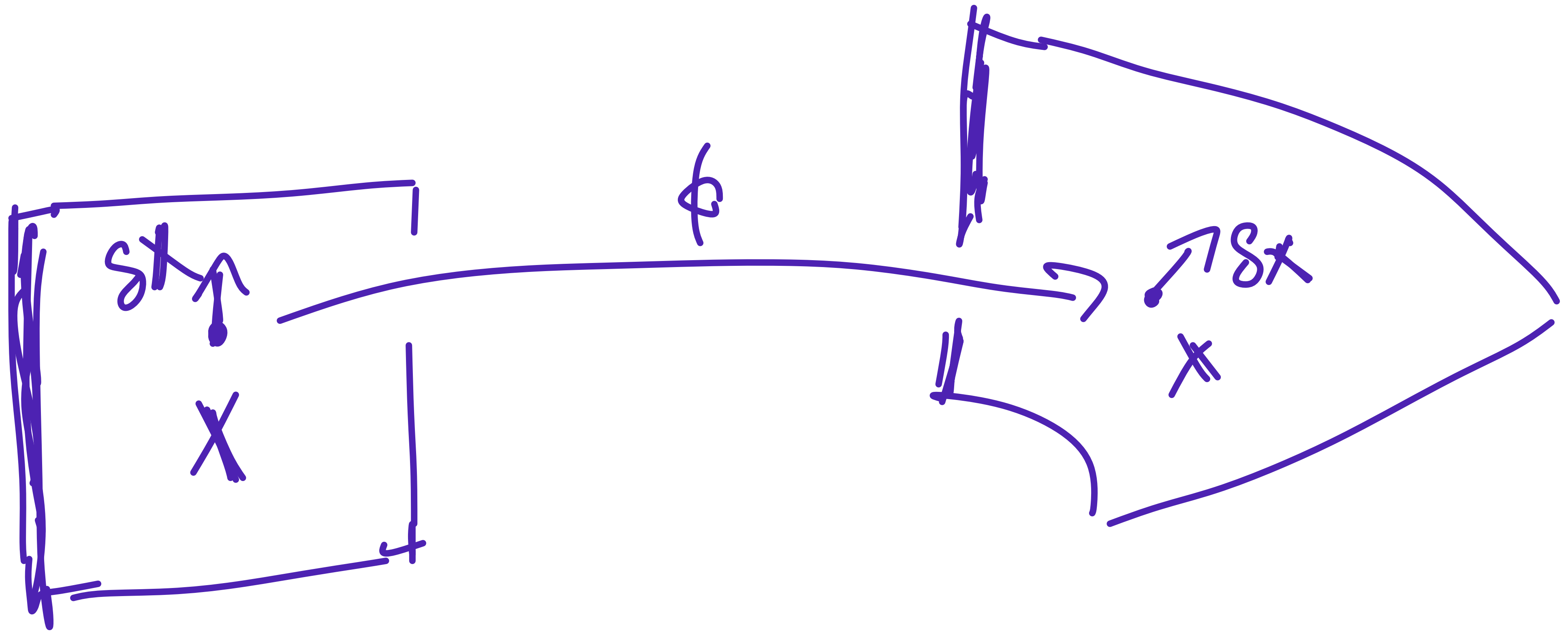
Deformation gradient

The material of the deformable object "wants" to return to the rest shape

- how do we describe this behavior exactly?
- bits of material can't communicate at a distance or "know" where they are in space
- all interactions are *local* — the motion of a point depends only on its neighborhood

Result: deformation models are based only on the *derivative* of ϕ

- $\mathbf{F} = \dfrac{\partial \phi}{\partial \mathbf{X}} = \dfrac{\partial \mathbf{x}}{\partial \mathbf{X}}$ or $\delta \mathbf{x} = \mathbf{F}\,\delta \mathbf{X}$
- $\mathbf{F}$ is a matrix—2x2 or 3x3 depending on the dimension of the simulation



Computing deformation gradient

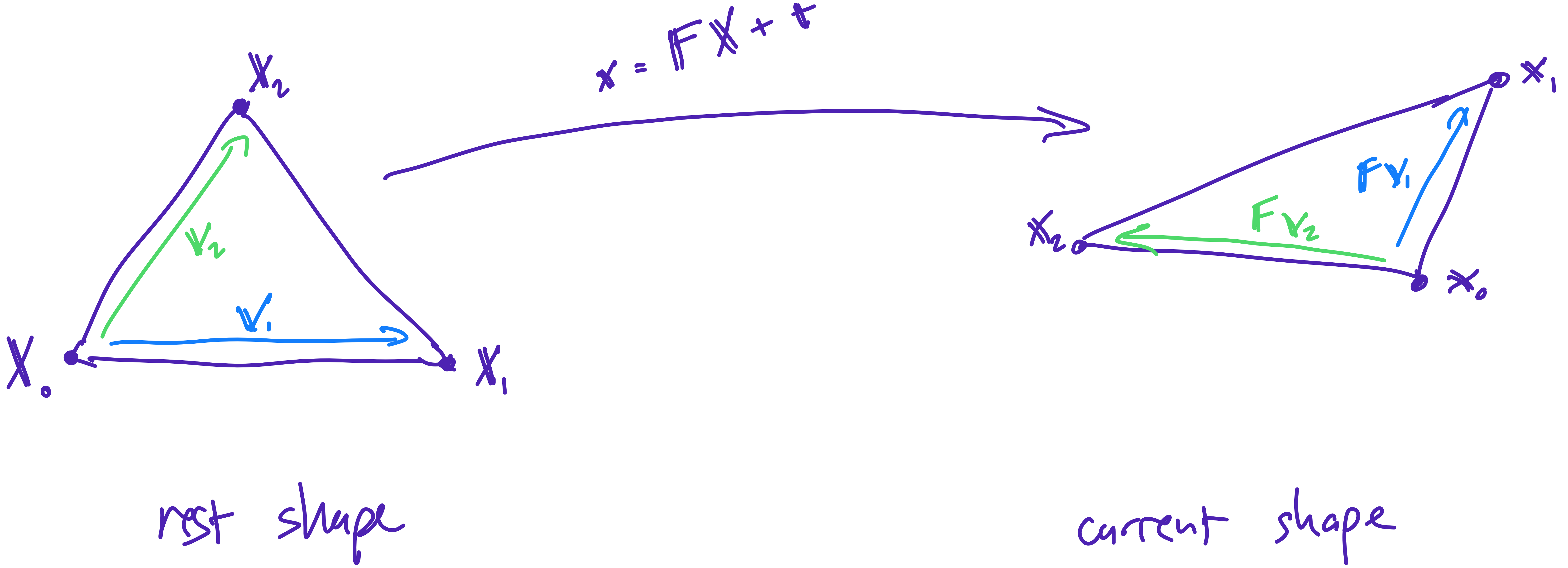
This is all very abstract — how do I compute it for a deforming mesh?

- very much like the computation used to get tangent vectors on a surface for shading
- in 2D, a triangle defines a unique affine map; in 3D a tetrahedron does the same
- can get that linear map by looking at triangle edge vectors

$$[\mathbf{x}_1 - \mathbf{x}_0 \quad \mathbf{x}_2 - \mathbf{x}_0] = \mathbf{F}\left[\mathbf{X}_1 - \mathbf{X}_0 \quad \mathbf{X}_2 - \mathbf{X}_0\right]$$

$$\mathbf{D} = \mathbf{F}\mathbf{D}_0$$

$$\mathbf{F} = \mathbf{D}\mathbf{D}_0^{-1}$$



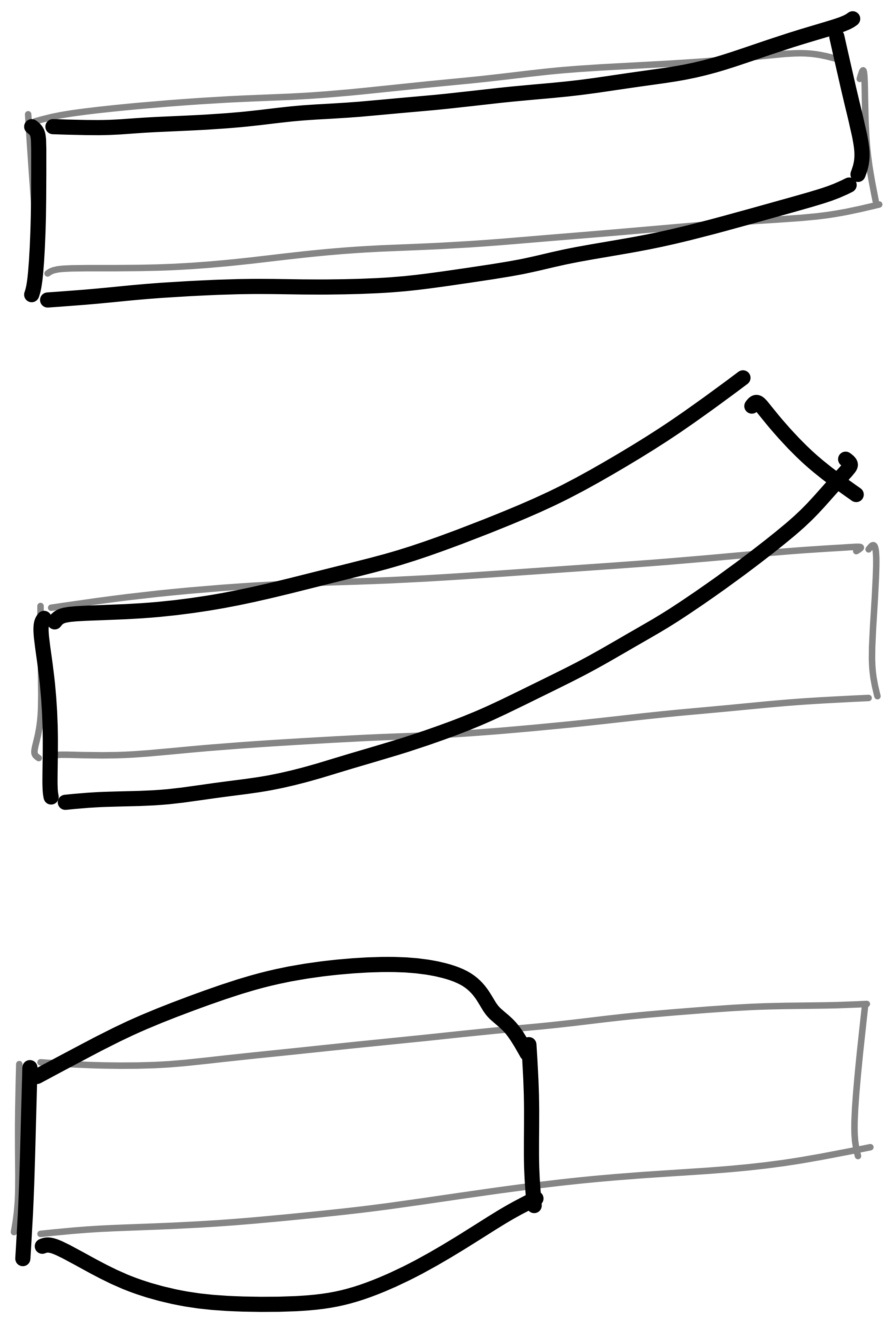
Infinitesimal vs. finite

When formulating elasticity problems there are multiple branches

- when things change just a bit from the rest config, linearized models are good
- when things change a lot, linearized models are very much not good

Two cases to distinguish

- small (infinitesimal) displacements →
    - the deformation map (and gradient) is close to the identity
    - the deformation map (and gradient) can be approximated with a linear model
- small (infinitesimal) strains →
    - the deformation gradient is close to rigid
    - the deformation gradient can be approximated with a linear model in the appropriate coordinates



Rotation invariance

Behavior of deformable model should be the same in all coordinate systems

- translation invariant — that is guaranteed by building on $\mathbf{F}$
- rotation invariant — rotating the object changes $\mathbf{F}$ but should not change behavior

Look at the SVD of $\mathbf{F}$ for insight

$$\mathbf{F} = \mathbf{U}\Sigma\mathbf{V}^T = [\mathbf{u}_1 \quad \mathbf{u}_2] \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} [\mathbf{v}_1 \quad \mathbf{v}_2]^T = \mathbf{R}_{\text{world}} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \mathbf{R}_{\text{material}}$$

- measures of deformation should not depend on $\mathbf{R}_{\text{world}}$

Isotropic material: material has no special orientation

- in this case quantities like energy should be independent of both $\mathbf{R}$s
- the key information about deformation is contained just in the σ_is

Hyperelastic materials

Elastic deformation: the material springs back to its original shape

Plastic deformation: the material changes internally and remains deformed

The idealization of a material that is elastic for all deformations is *hyperelastic*

Hyperelastic materials:

- deform without losing energy
- can be entirely described using a potential energy: *strain energy*
- strain energy is analogous to the familiar $\frac{1}{2}kx^2$ potential for linear springs
- strain energy is the integral of a volume density: *strain energy density*
- for homogeneous materials there is a single function ψ that computes strain energy density from $\mathbf{F}$

$$E[\psi] = \int_B \psi(\mathbf{F}(\mathbf{X}))\, d\mathbf{X}$$

Measuring strain

Strain measures

- functions of deformation gradient $\mathbf{F}$
- should be zero for $\mathbf{F} = \mathbf{I}$
- should be rotation-invariant in the world (for large displacements)
- looking at SVD $\mathbf{F} = \mathbf{U\Sigma V}^T$, strain should be independent of $\mathbf{U}$

Two routes to rotation invariance

- use a product to cancel $\mathbf{U}$: $\mathbf{F}^T\mathbf{F} = \mathbf{V\Sigma U}^T\mathbf{U\Sigma V}^T = \mathbf{V\Sigma}^2\mathbf{V}^T$ (this is the “right Cauchy-Green deformation tensor”)
- use a matrix decomposition to separate out rotation:
  - compute the polar decomposition: $\mathbf{F} = \mathbf{RS} = (\mathbf{UV}^T)(\mathbf{V\Sigma V}^T)$ and measure strain from just $\mathbf{S}$

Three basic strain measures

Green's strain: $\mathbf{E}(\mathbf{F}) = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right)$

- simple to compute
- rotation invariant in world
- ...but measures the square of the stretch factor
  - $\mathbf{E} = \frac{1}{2}\left(\mathbf{V}\mathbf{\Sigma}^2\mathbf{V}^T - \mathbf{I}\right) = \frac{1}{2}\left(\mathbf{V}\mathbf{\Sigma}^2\mathbf{V}^T - \mathbf{V}\mathbf{V}^T\right) = \mathbf{V}\left(\frac{1}{2}(\mathbf{\Sigma}^2 - \mathbf{I})\right)\mathbf{V}^T$

Corotated linear strain: $\epsilon_c = \mathbf{S} - \mathbf{I}$

- "corotated" meaning computed in a coordinate system that rotates with the object
- strain defined based only on the $\mathbf{S}$ factor from the polar decomposition (ignore $\mathbf{R}$)
- measures the stretch factor directly
  - $\mathbf{E} = \mathbf{V}\mathbf{\Sigma}\mathbf{V}^T - \mathbf{V}\mathbf{V}^T = \mathbf{V}(\mathbf{\Sigma} - \mathbf{I})\mathbf{V}^T$

Aside: how it plays out in 1D

A 1D deformable object living in a 1D space

- no rotation, no distinction between deformation and strain
- deformation map is just a function $x = \phi(X) : \mathbb{R} \to \mathbb{R}$
- deformation gradient is its derivative $F(X) = \dfrac{d\phi}{dX} = \phi'(X)$
- strain is measuring the deviation of F from 1
- linear strain: $\epsilon = F - 1$
- Green's strain: $E = \frac{1}{2}\left(F^2 - 1\right)$
- these match for small strains (near $F = 1$) but diverge as strain increases

Linear algebra aside

Frobenius norm

- a measure of “size” for matrices
- amounts to thinking of the $N \times N$ matrix as a N^2-vector and using the Euclidean norm
  - $\|\mathbf{A}\|_F^2 = \sum_{i,j} a_{ij}^2$
- rotation invariance: F-norm is invariant to rotation on either side
  - proof: think of matrix as a stack of columns or rows

$$\mathbf{A} = [\mathbf{v}_1 \cdots \mathbf{v}_n]$$

$$\mathbf{QA} = \left[\mathbf{Qv}_1 \cdots \mathbf{Qv}_n\right]$$

$$\|\mathbf{A}\|_F^2 = \sum_k \|\mathbf{v}\|_2^2 = \sum_k \|\mathbf{Qv}\|_2^2$$

Linear algebra aside

Double contraction aka. “double dot product”

- like a dot product operation for matrices: $\mathbf{A} : \mathbf{B} = \sum_{i,j} a_{ij} b_{ij}$
- leads to another way to write the F-norm: $\|\mathbf{A}\|_F^2 = \mathbf{A} : \mathbf{A}$
- handy identities:
  - $\mathbf{A} : \mathbf{BC} = \mathbf{B}^T\mathbf{A} : \mathbf{C} = \mathbf{AC}^T : \mathbf{B}$
  - $\delta[\mathbf{A} : \mathbf{B}] = \delta[\mathbf{A}] : \mathbf{B} + \mathbf{A} : \delta[\mathbf{B}]$
  - $\delta[\|\mathbf{A}\|_F^2] = \delta[\mathbf{A} : \mathbf{A}] = 2\mathbf{A} : \delta\mathbf{A}$

More matrix invariants

Invariants = functions that are invariant to change of basis

- Frobenius norm is an invariant

Trace of matrix: sum of diagonal elements

- $\operatorname{tr}\mathbf{A} = \sum_i a_{ii}$ another way to write this: $\operatorname{tr}\mathbf{A} = \mathbf{I} : \mathbf{A}$
- useful facts: $\operatorname{tr}\mathbf{A} = \operatorname{tr}\mathbf{A}^T$; $\operatorname{tr}\mathbf{AB} = \operatorname{tr}\mathbf{BA}$; $\operatorname{tr}\mathbf{A}^T\mathbf{B} = \mathbf{A} : \mathbf{B}$ (prove by writing out the sums)
- corollary: $\operatorname{tr}\mathbf{QAQ}^T = \operatorname{tr}(\mathbf{AQ}^T)\mathbf{Q} = \operatorname{tr}\mathbf{A}$
- for symmetric matrices $\mathbf{A} = \mathbf{V\Sigma V}^T$ and $\operatorname{tr}\mathbf{A} = \operatorname{tr}\mathbf{\Sigma} = \sum_i \sigma_i$

More matrix invariants

Determinant of matrix: (signed) volume spanned by columns

- determinant tells how much the transformation magnifies area or volume
- useful facts: $\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}$; $\det \mathbf{A} = \det \mathbf{A}^T$
- corollary: $\det \mathbf{QA} = 1 \det \mathbf{A} = \det \mathbf{A}$ — determinant invariant to rotations on both sides
- since $\mathbf{A} = \mathbf{U\Sigma V}^T$, $\det \mathbf{A} = \det \mathbf{\Sigma} = \prod_i \sigma_i$

So all together we have three invariants that are easy to compute

- Frobenius norm: $\|\mathbf{A}\|_F^2$ is the sum of squares of the singular values
- trace: $\operatorname{tr} \mathbf{A}$ is the sum of the singular values for symmetric $\mathbf{A}$ (which will be the case for us)
- determinant: $\det \mathbf{A}$ is the product of the singular values

Constitutive models

For hyperelastic materials, just need to define a strain energy density

- function of strain at a point
- for isotropic materials, should be rotation invariant in the material space
- this means they ultimately are just functions of the singular values of strain
- typically they are defined as simple functions of the invariants on the previous slide

Three basic linear models

- Linear elasticity
- St. Venant-Kirchoff model
- Corotated linear elasticity
- they all define ψ in the same way, but they start with the three different strain measures

Properties of elastic materials

Materials are described in terms of observable macroscopic properties

- take a block of material, apply uniaxial tension or compression
- object behaves like a spring (pushes back proportional to displacement)
- spring constant is proportional to cross-section:
  - $f = k\Delta L$; $k = AE/L$
  - E is known as Young's modulus (force/area)
- material also changes along the other axis (aka. laterally)
  - most materials resist changing volume
  - with no lateral force, lateral shrinkage is proportional to axial extension
  - $\Delta w = -\nu\Delta L$; ν is known as Poisson's ratio
  - in 3D $\nu = 0.5$ is exact volume preservation (in 2D, corresponding parameter is $\nu = 1$)

Linear elasticity

Simplest model for this small-deformation behavior

- make energy a linear combination of the two easiest-to-compute invariants
- first think about just E, assuming $\nu = 0$
- want spring energy to be $\frac{1}{2}k(\Delta L)^2$, so energy/volume is $\frac{1}{2}E\epsilon_l^2$
- $\psi(\mathbf{F}) = \mu\|\boldsymbol{\epsilon}\|_F^2 = \mu\epsilon_l^2$; $\mu = E/2$ where ϵ_l is lengthwise strain and there is no transverse strain
- to account for ν as well, add a term for the trace
- $\psi(\mathbf{F}) = \mu\|\boldsymbol{\epsilon}\|_F^2 + \dfrac{\lambda}{2}(\operatorname{tr}\boldsymbol{\epsilon})^2$
- if you solve for μ and λ to provide the same energy when $\epsilon_t = -\nu\epsilon_l$ you get the formulas:
- $\mu = \dfrac{E}{2(1+\nu)}$ and $\lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)}$ (in 3D) or $\lambda = \dfrac{E\nu}{(1+\nu)(1-\nu)}$ (in 2D)

Linear strain

For small deformations we can use a first-order approximation to $\mathbf{E}$

- $\mathbf{E}(\mathbf{F}) \approx \mathbf{E}(\mathbf{I}) + \delta\mathbf{E}(\mathbf{I})\,|_{\delta\mathbf{F}=\mathbf{F}-\mathbf{I}} + \ldots$

$$\mathbf{E}(\mathbf{F}) \approx \delta\mathbf{E}(\mathbf{I})\,|_{\delta\mathbf{F}=\mathbf{F}-\mathbf{I}}$$

$$= \frac{1}{2}\left((\mathbf{F}-\mathbf{I})^T + (\mathbf{F}-\mathbf{I})\right)$$

- $$= \frac{1}{2}\left(\mathbf{F}+\mathbf{F}^T\right) - \mathbf{I}$$

$$\delta\mathbf{E}(\mathbf{F}) = \frac{1}{2}\delta[\mathbf{F}^T\mathbf{F}-\mathbf{I}]$$

$$= \frac{1}{2}\left(\delta[\mathbf{F}]^T\mathbf{F} + \mathbf{F}^T\delta[\mathbf{F}]\right)$$

$$\delta\mathbf{E}(\mathbf{I}) \approx \frac{1}{2}\left(\delta[\mathbf{F}]^T\mathbf{I} + \mathbf{I}^T\delta[\mathbf{F}]\right)$$

$$= \frac{1}{2}\left(\delta[\mathbf{F}]^T + \delta[\mathbf{F}]\right)$$

infinitesimal (linear) strain: $\epsilon = \frac{1}{2}\left(\mathbf{F}+\mathbf{F}^T\right) - \mathbf{I}$

Linear elasticity

The first constitutive model for an isotropic material

- measure deformation using the linear strain $\boldsymbol{\epsilon} = \frac{1}{2}\left(\mathbf{F} + \mathbf{F}^T\right) - \mathbf{I}$
- define strain energy density as $\psi = \mu \|\boldsymbol{\epsilon}\|_F^2 + \frac{\lambda}{2}(\operatorname{tr} \boldsymbol{\epsilon})^2$

To determine forces on mesh vertices we need $\partial\psi / \partial\mathbf{x}_i$

- the chain-rule chain is $\mathbf{x} \rightarrow \mathbf{F} \rightarrow \boldsymbol{\epsilon} \rightarrow \psi \rightarrow E$
- we already derived $\partial\mathbf{F}/\partial\mathbf{x}$ and $\partial E/\partial\psi$ will be simple
- we still need $\partial\boldsymbol{\epsilon}/\partial\mathbf{F}$ and $\partial\psi/\partial\boldsymbol{\epsilon}$ (these are the ones that depend on the material model)
- will work these two out using variational notation and derive a formula for

$$\mathbf{P}(\mathbf{F}) = \frac{\partial\psi(\mathbf{F})}{\partial\mathbf{F}}$$ known as the “first Piola-Kirchoff stress”

Energy density gradient for linear elasticity

First the derivative of elastic energy density with respect to strain

- $$\begin{aligned}\delta\psi(\mathbf{F}) &= \delta[\mu\|\boldsymbol{\epsilon}(\mathbf{F})\|_F^2 + \frac{\lambda}{2}(\operatorname{tr}\boldsymbol{\epsilon}(\mathbf{F}))^2] \\ &= 2\mu\boldsymbol{\epsilon} : \delta\boldsymbol{\epsilon} + \lambda(\operatorname{tr}\boldsymbol{\epsilon})\mathbf{I} : \delta\boldsymbol{\epsilon}\end{aligned}$$

next the derivative of strain with respect to deformation gradient

- $\boldsymbol{\epsilon} = \frac{1}{2}\left(\mathbf{F} + \mathbf{F}^T\right) - \mathbf{I}$
- $\delta\boldsymbol{\epsilon} = \delta[\operatorname{sym}\mathbf{F}] = \operatorname{sym}\delta\mathbf{F}$ where $\operatorname{sym}\mathbf{A} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right)$

substituting:

- $\delta\psi(\mathbf{F}) = \left(2\mu\boldsymbol{\epsilon} + \lambda(\operatorname{tr}\boldsymbol{\epsilon})\mathbf{I}\right) : \delta\mathbf{F}$ —simplified via $\mathbf{S} : \operatorname{sym}\mathbf{A} = \mathbf{S} : \mathbf{A}$ for symmetric $\mathbf{S}$
- $\mathbf{P} = 2\mu\boldsymbol{\epsilon} + \lambda(\operatorname{tr}\boldsymbol{\epsilon})\mathbf{I}$ —this is $\partial\psi/\partial\mathbf{F}$

Computing nodal forces

Now we have the complete chain of derivatives for the first model

- let's compute the forces as $\mathbf{f}_i = -\,\partial E / \partial \mathbf{x}_i$
- $E[\phi] = \int_B \psi(\mathbf{F}(\mathbf{X}))\, d\mathbf{X}$
- $E[\phi] = \sum_k \int_{T_k} \psi(\mathbf{F}(\mathbf{X}))\, d\mathbf{X} = \sum_k |T_k| \psi(\mathbf{F}_k)$
- recall that $\mathbf{F} = \mathbf{D}\mathbf{D}_0^{-1}$, so $\delta\mathbf{F} = \delta[\mathbf{D}]\,\mathbf{D}_0^{-1}$
- we have $\delta\psi = \mathbf{P} : \delta\mathbf{F}$, so $\delta\psi = \mathbf{P} : (\delta[\mathbf{D}]\,\mathbf{D}_0^{-1}) = \mathbf{P}\mathbf{D}_0^{-T} : \delta\mathbf{D}$
- for triangle k, $\delta E = |T_k| \delta\psi = |T_k| \mathbf{P}\mathbf{D}_0^{-T} : \delta\mathbf{D}$
- thus $\delta E = -\,\mathbf{H} : \delta\mathbf{D}$ where $\mathbf{H} = -\,|T_k| \mathbf{P}\mathbf{D}_0^{-T}$
- $\mathbf{H} = \begin{bmatrix}\mathbf{f}_1 & \mathbf{f}_2\end{bmatrix}$ and $\mathbf{f}_0 = -\,(\mathbf{f}_1 + \mathbf{f}_2)$

To sum up…

So you are writing a simulator and need to compute forces on your vertices?

No problem, follow these 5 steps:

To compute the forces due to one triangle:

1. Ahead of time compute $\mathbf{D}_0^{-1}$ and $|T_k|$.
2. Compute $\mathbf{F} = \mathbf{D}\mathbf{D}_0^{-1}$ from the current vertex positions.
3. Compute the strain from $\mathbf{F}$ using the formulas appropriate to your model.
4. Compute the stress $\mathbf{P}$ from the strain using the formulas appropriate to your model.
5. Compute $\mathbf{H} = -|T_k|\mathbf{P}\mathbf{D}_0^{-T}$, and the forces are sitting in the columns of $\mathbf{H}$.

To compute the total force on each vertex you need to loop over all the triangles
and accumulate their contributions. That's all there is to it!

Forces for nonlinear models

Two other models are commonly used that are built similarly to linear elasticity

St. Venant–Kirchoff model

- based on Green's strain: $\mathbf{E}(\mathbf{F}) = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right)$
- uses same strain energy density formula as linear elasticity: $\psi = \mu\|\mathbf{E}\|_F^2 + \frac{\lambda}{2}(\operatorname{tr}\mathbf{E})^2$
- to differentiate energy, the first step is the same as linear: $\delta\psi = 2\mu\mathbf{E} : \delta\mathbf{E} + \lambda(\operatorname{tr}\mathbf{E})\mathbf{I} : \delta\mathbf{E}$
- to differentiate strain, $\delta\mathbf{E} = \operatorname{sym}\left(\mathbf{F}^T\delta\mathbf{F}\right)$
- remember $\mathbf{S} : \operatorname{sym}\mathbf{A} = \mathbf{S} : \mathbf{A}$ and $\mathbf{E}$ and $\mathbf{I}$ are symmetric. so:
- $\delta\psi = 2\mu\mathbf{F}\mathbf{E} : \delta\mathbf{F} + \lambda(\operatorname{tr}\mathbf{E})\mathbf{F} : \delta F = \mathbf{F}\left(2\mu\mathbf{E} + \lambda(\operatorname{tr}\mathbf{E})\mathbf{I}\right) : \delta F$
- read off $\mathbf{P} = \mathbf{F}\left[2\mu\mathbf{E} + \lambda(\operatorname{tr}\mathbf{E})\mathbf{I}\right]$

Forces for nonlinear models

Corotated linear model

- based on corotated strain: $\boldsymbol{\epsilon}_c = \mathbf{S} - \mathbf{I}$ where $\mathbf{F} = \mathbf{RS}$; $\psi = \mu \|\boldsymbol{\epsilon}_c\|_F^2 + \frac{\lambda}{2}(\mathrm{tr}\, \boldsymbol{\epsilon}_c)^2$
- to differentiate energy, $\delta\psi = 2\mu\boldsymbol{\epsilon}_c : \delta\boldsymbol{\epsilon}_c + \lambda(\mathrm{tr}\, \boldsymbol{\epsilon}_c)\mathbf{I} : \delta\boldsymbol{\epsilon}_c$ and $\delta\boldsymbol{\epsilon}_c = \delta\mathbf{S}$
- lemma: $\delta\mathbf{S} = \mathbf{R}^T\delta\mathbf{F} - \left(\mathbf{R}^T\delta\mathbf{R}\right)\mathbf{S}$ (at right)
- lemma: $\mathbf{R}^T\delta\mathbf{R}$ is antisymmetric (at right)
- then substituting above:

$$\delta\psi = \left(2\mu\boldsymbol{\epsilon}_c + \lambda(\mathrm{tr}\, \boldsymbol{\epsilon}_c)\mathbf{I}\right) : \mathbf{R}^T\delta\mathbf{F} + \left(2\mu\boldsymbol{\epsilon}_c + \lambda(\mathrm{tr}\, \boldsymbol{\epsilon}_c)\mathbf{I}\right) : \left(\mathbf{R}^T\delta\mathbf{R}\right)\mathbf{S}$$

$$= \mathbf{R}\left(2\mu\boldsymbol{\epsilon}_c + \lambda(\mathrm{tr}\, \boldsymbol{\epsilon}_c)\mathbf{I}\right) : \delta\mathbf{F} + \left(2\mu\boldsymbol{\epsilon}_c\mathbf{S} + \lambda(\mathrm{tr}\, \boldsymbol{\epsilon}_c)\mathbf{S}\right) : \left(\mathbf{R}^T\delta\mathbf{R}\right)$$

-

$$= \mathbf{R}\left(2\mu\boldsymbol{\epsilon}_c + \lambda(\mathrm{tr}\, \boldsymbol{\epsilon}_c)\mathbf{I}\right) : \delta\mathbf{F}$$

- read off $\mathbf{P} = \mathbf{R}\left[2\mu\boldsymbol{\epsilon}_c + \lambda(\mathrm{tr}\, \boldsymbol{\epsilon}_c)\mathbf{I}\right]$

$$\delta\mathbf{F} = \delta[\mathbf{R}]\mathbf{S} + \mathbf{R}\delta[\mathbf{S}]$$

$$\mathbf{R}\delta[\mathbf{S}] = \delta\mathbf{F} - \delta[\mathbf{R}]\mathbf{S}$$

$$\delta\mathbf{S} = \mathbf{R}^T\delta\mathbf{F} - \left(\mathbf{R}^T\delta\mathbf{R}\right)\mathbf{S}$$

$$\delta(\mathbf{R}^T\mathbf{R}) = 0 = \delta\mathbf{R}^T\mathbf{R} + \mathbf{R}^T\delta\mathbf{R} = \mathbf{R}^T\delta\mathbf{R} + \left(\mathbf{R}^T\delta\mathbf{R}\right)^T$$

One more nonlinear model

To be useful for significant compression, push back against $\det \mathbf{F}$

- the determinant measures volume change accurately for large strains
- incorporating the logarithm of $\det \mathbf{F}$ in the energy makes it diverge as volume $\to 0$
- a widely used *neo-Hookean* model is: $\psi(\mathbf{F}) = \frac{\mu}{2}(\|\mathbf{F}\|_F^2 - 3) - \mu \log \det \mathbf{F} + \frac{\lambda}{2}(\log \det \mathbf{F})^2$
- to differentiate this, make use of Jacobi's formula $\delta[\det \mathbf{A}] = (\det \mathbf{A})\, \mathbf{A}^{-T} : \delta \mathbf{A}$
- omitting a few details, the three terms in $\delta\psi$ are:
  - $\frac{\mu}{2}\delta[\|\mathbf{F}\|_F^2] = \mu\mathbf{F} : \delta\mathbf{F}$; $\mu\delta[\log \det \mathbf{F}] = \mu\mathbf{F}^{-T} : \delta\mathbf{F}$; $\frac{\lambda}{2}\delta\left[(\log \det \mathbf{F})^2\right] = \lambda(\log \det \mathbf{F})\, \mathbf{F}^{-T} : \delta\mathbf{F}$
- end result $\mathbf{P}(\mathbf{F}) = \mu(\mathbf{F} - \mathbf{F}^{-T}) + \lambda(\log \det \mathbf{F})\, \mathbf{F}^{-T}$