Deformation models
Working out our spring force from the energy

Start with the spring energy

\[ E_{ij}(\mathbf{x}) = \frac{1}{2} k_s (\|\mathbf{x}_i - \mathbf{x}_j\| - l_0)^2 \] (this is the contribution of one spring to the total system energy)

Force is minus the gradient of energy

\[ \mathbf{f}_i(\mathbf{x}) = -\frac{\partial E}{\partial \mathbf{x}_i}(\mathbf{x}) \] (remember \( \mathbf{x} \) is a big vector of all the positions; this partial derivative is zero for all the particles that are not connected to this particular spring)

Take the computation one step at a time:

- derivative of \( \mathbf{x}_i - \mathbf{x}_j \) is \( I \) wrt. \( \mathbf{x}_i \) and \(-I \) wrt. \( \mathbf{x}_j \)
- derivative of \( \|\mathbf{v}\| \) wrt. \( \mathbf{v} \) is \( \hat{\mathbf{v}} \)
- derivative of \( E_{ij} \) wrt \( \|\mathbf{v}\| \) is \( k_s (\|\mathbf{v}\| - l_0) \)
- put it all together: \( \mathbf{f}_i = -\partial E/\partial \mathbf{x}_i = -k_s (\|\mathbf{x}_i\| - l_0) \hat{\mathbf{x}}_{ij} \) and \( \mathbf{f}_j = -\partial E/\partial \mathbf{x}_j = k_s (\|\mathbf{x}_{ij}\| - l_0) \hat{\mathbf{x}}_{ij} \) where \( \mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j \)
Alternative “variational” notation

Derivative is a linear transformation; write down the output

• instead of \( \frac{\partial f}{\partial x} = A \) write \( \delta f = A \delta x \)

• when the matrix \( A \) is awkward to write down this can be neater...

• \( \delta x_{ij} = \delta x_i - \delta x_j \)

• \( \delta \|v\| = \hat{v} \cdot \delta v \)

• \( \delta E = k_s(l - l_0)\delta l \)

• substitute to get \( \delta E = k_s(\|x_{ij}\| - l_0)\hat{x}_{ij} \cdot \delta x_i - k_s(\|x_{ij}\| - l_0)\hat{x}_{ij} \cdot \delta x_j \)

• read off \( f_i \) and \( f_j \)
Deformable models

Mass-spring models can get you somewhere
  • but only so far
  • they were used a lot back in the Old Days

They have their limitations
  • hard to separate different stiffnesses (e.g. bend/shear springs contribute to stretch)
  • hard to control preservation of volume in deformations
  • hard to make them agree with measurements

Let’s keep the idea of deriving forces from energies
  • define energies to get the behavior we want
  • borrow energies from other fields to get more accurate models
Example: hinge energy

We made a rope before using linear springs

- connect springs between every other point
- when rope bends, the springs fight one another, indirectly causing bending resistance

More direct approach

- just make the energy depend on the bending angle \( \theta \) (well, \( \sin \frac{\theta}{2} \))

\[
E = k \sin \frac{\theta}{2} = \frac{k}{2} (-\cos \theta) \quad \text{equiv.} \quad E = -\frac{k}{2} \cos \theta.
\]

\[
\cos \theta = \hat{x}_{12} \cdot \hat{x}_{23}
\]

\[
\delta \cos \theta = \frac{1}{\| \delta x_{12} \|} \left( \hat{x}_{23} - (\hat{x}_{23} \cdot \hat{x}_{12}) \hat{x}_{12} \right) \cdot \delta x_{12} + \frac{1}{\| \delta x_{23} \|} \left( \hat{x}_{12} - (\hat{x}_{12} \cdot \hat{x}_{23}) \hat{x}_{23} \right) \cdot \delta x_{23}
\]

\[
\delta (\hat{x} \cdot \delta \hat{x}) = \delta \hat{x} \cdot (\delta \hat{x})^T + \text{cyclic sym.}
\]

\[
= \frac{1}{\| \delta x \|} \left( \delta \hat{x} \cdot (\delta \hat{x})^T \right) \delta x + \frac{1}{\| \delta \hat{x} \|} \left( (\delta \hat{x})^T \cdot (\delta \hat{x}) \right) \delta \hat{x}
\]
A deforming object is described by a time varying function

\[ x = \phi(X, t) \]

- maps the rest position of a chunk of material to its current deformed position
- aka. a map from material space to world space
- varies as a function of time
The material of the deformable object “wants” to return to the rest shape

- how do we describe this behavior exactly?
- bits of material can’t communicate at a distance or “know” where they are in space
- all interactions are local — the motion of a point depends only on its neighborhood

Result: deformation models are based only on the derivative of $\phi$

$$F = \frac{\partial \phi}{\partial X} = \frac{\partial x}{\partial X} \text{ or } \delta x = F \delta X$$

- $F$ is a matrix — 2x2 or 3x3 depending on the dimension of the simulation
Computing deformation gradient

This is all very abstract — how do I compute it for a deforming mesh?

• very much like the computation used to get tangent vectors on a surface for shading
• in 2D, a triangle defines a unique affine map; in 3D a tetrahedron does the same
• can get that linear map by looking at triangle edge vectors

\[
\begin{bmatrix}
  x_1 - x_0 \\
  x_2 - x_0
\end{bmatrix} = F \begin{bmatrix}
  X_1 - X_0 \\
  X_2 - X_0
\end{bmatrix}
\]

\[
D = FD_0 \\
F = DD_0^{-1}
\]
Infinitesimal vs. finite

When formulating elasticity problems there are multiple branches

• when things change just a bit from the rest config, linearized models are good
• when things change a lot, linearized models are very much not good

Two cases to distinguish

• small (infinitesimal) displacements →
  - the deformation map (and gradient) is close to the identity
  - the deformation map (and gradient) can be approximated with a linear model

• small (infinitesimal) strains →
  - the deformation gradient is close to rigid
  - the deformation gradient can be approximated with a linear model in the appropriate coordinates
Rotation invariance

Behavior of deformable model should be the same in all coordinate systems

- translation invariant — that is guaranteed by building on $F$
- rotation invariant — rotating the object changes $F$ but should not change behavior

Look at the SVD of $F$ for insight

$$F = U \Sigma V^T = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix} = R_{\text{world}} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} R_{\text{material}}$$

- measures of deformation should not depend on $R_{\text{world}}$

Isotropic material: material has no special orientation

- in this case quantities like energy should be independent of both $R$s
- the key information about deformation is contained just in the $\sigma_i$s
Hyperelastic materials

Elastic deformation: the material springs back to its original shape

Plastic deformation: the material changes internally and remains deformed

The idealization of a material that is elastic for all deformations is *hyperelastic*

Hyperelastic materials:

- deform without losing energy
- can be entirely described using a potential energy: *strain energy*
- strain energy is analogous to the familiar $\frac{1}{2}kx^2$ potential for linear springs
- strain energy is the integral of a volume density: *strain energy density*

for homogeneous materials there is a single function $\psi$

that computes strain energy density from $F$

$$E[\psi] = \int_B \psi(F(X)) \, dX$$
Measuring strain

Strain measures

- functions of deformation gradient $F$
- should be zero for $F = I$
- should be rotation-invariant in the world (for large displacements)
- looking at SVD $F = U \Sigma V^T$, strain should be independent of $U$

Two routes to rotation invariance

- use a product to cancel $U$: $F^T F = V \Sigma U^T U \Sigma V^T = V \Sigma^2 V^T$ (this is the “right Cauchy-Green deformation tensor”)
- use a matrix decomposition to separate out rotation:
  - compute the polar decomposition: $F = RS = (UV^T)(V \Sigma V^T)$
  and measure strain from just $S$
Three basic strain measures

**Green's strain:** \( E(F) = \frac{1}{2} (F^T F - I) \)
- simple to compute
- rotation invariant in world
- ...but measures the square of the stretch factor

\[
E = \frac{1}{2} (V \Sigma^2 V^T - I) = \frac{1}{2} (V \Sigma^2 V^T - V V^T) = V \left( \frac{1}{2} (\Sigma^2 - I) \right) V^T
\]

**Corotated linear strain:** \( \sigma_c = S - I \)
- "corotated" meaning computed in a coordinate system that rotates with the object
- strain defined based only on the \( S \) factor from the polar decomposition (ignore \( R \))
- measures the stretch factor directly

\[
E = V \Sigma V^T - V V^T = V (\Sigma - I) V^T
\]
Aside: how it plays out in 1D

**A 1D deformable object living in a 1D space**

- no rotation, no distinction between deformation and strain
- deformation map is just a function $x = \phi(X) : \mathbb{R} \rightarrow \mathbb{R}$
  - deformation gradient is its derivative $F(X) = \frac{d\phi}{dX} = \phi'(X)$
- strain is measuring the deviation of $F$ from 1
- linear strain: $\epsilon = F - 1$
- Green’s strain: $E = \frac{1}{2} \left( F^2 - 1 \right)$
- these match for small strains (near $F = 1$) but diverge as strain increases
Linear algebra aside

**Frobenius norm**

- a measure of “size” for matrices
- amounts to thinking of the $N \times N$ matrix as a $N^2$-vector and using the Euclidean norm
  \[ \|A\|_F^2 = \sum_{i,j} a_{ij}^2 \]
- rotation invariance: F-norm is invariant to rotation on either side
  - proof: think of matrix as a stack of columns or rows

\[
A = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \\
QA = \begin{bmatrix} Qv_1 & \cdots & Qv_n \end{bmatrix} \\
\|A\|_F^2 = \sum_k \|v\|_2^2 = \sum_k \|Qv\|_2^2
\]
Linear algebra aside

Double contraction aka. "double dot product"

- like a dot product operation for matrices: $A : B = \sum_{i,j} a_{ij} b_{ij}$
- leads to another way to write the F-norm: $\|A\|_F^2 = A : A$
- handy identities:
  - $A : BC = B^T A : C = AC^T : B$
  - $\delta[A : B] = \delta[A] : B + A : \delta[B]$
  - $\delta[\|A\|_F^2] = \delta[A : A] = 2A : \delta A$
More matrix invariants

Invariants = functions that are invariant to change of basis

- Frobenius norm is an invariant

Trace of matrix: sum of diagonal elements

\[ \text{tr } A = \sum_i a_{ii} \quad \text{another way to write this: } \text{tr } A = I : A \]

- useful facts: \( \text{tr } A = \text{tr } A^T \); \( \text{tr } AB = \text{tr } BA \); \( \text{tr } A^T B = A : B \) (prove by writing out the sums)
- corollary: \( \text{tr } QAQ^T = \text{tr } (AQ^T)Q = \text{tr } A \)
- for symmetric matrices \( A = V\Sigma V^T \) and \( \text{tr } A = \text{tr } \Sigma = \sum_i \sigma_i \)
More matrix invariants

**Determinant of matrix:** (signed) volume spanned by columns

- determinant tells how much the transformation magnifies area or volume
- useful facts: \( \det AB = \det A \det B \); \( \det A = \det A^T \)
- corollary: \( \det QA = 1 \) \( \det A = \det A \) — determinant invariant to rotations on both sides
  - since \( A = UΣV^T \), \( \det A = \det Σ = \prod_i σ_i \)

So all together we have three invariants that are easy to compute

- Frobenius norm: \( ||A||_F^2 \) is the sum of squares of the singular values
- trace: \( \text{tr } A \) is the sum of the singular values for symmetric \( A \) (which will be the case for us)
- determinant: \( \det A \) is the product of the singular values
Constitutive models

For hyperelastic materials, just need to define a strain energy density

- function of strain at a point
- for isotropic materials, should be rotation invariant in the material space
- this means they ultimately are just functions of the singular values of strain
- typically they are defined as simple functions of the invariants on the previous slide

Three basic linear models

- Linear elasticity
- St. Venant-Kirchhoff model
- Corotated linear elasticity
- they all define $\psi$ in the same way, but they start with the three different strain measures
Properties of elastic materials

Materials are described in terms of observable macroscopic properties

• take a block of material, apply uniaxial tension or compression
• object behaves like a spring (pushes back proportional to displacement)
• spring constant is proportional to cross-section:
  - \( f = k \Delta L \); \( k = AE/L \)
  - \( E \) is known as Young’s modulus (force/area)
• material also changes along the other axis (aka. laterally)
  - most materials resist changing volume
  - with no lateral force, lateral shrinkage is proportional to axial extension
  - \( \Delta w = -\nu \Delta L \); \( \nu \) is known as Poisson’s ratio
  - in 3D \( \nu = 0.5 \) is exact volume preservation
    (in 2D, corresponding parameter is \( \nu = 1 \))
Linear elasticity

Simplest model for this small-deformation behavior

• make energy a linear combination of the two easiest-to-compute invariants

• first think about just $E$, assuming $\nu = 0$

• want spring energy to be $\frac{1}{2}k(\Delta L)^2$, so energy/volume is $\frac{1}{2}E\epsilon_l^2$

• $\psi(F) = \mu \|\epsilon\|^2_F = \mu \epsilon_l^2$; $\mu = E/2$ where $\epsilon_l$ is lengthwise strain and there is no transverse strain

• to account for $\nu$ as well, add a term for the trace

  . $\psi(F) = \mu \|\epsilon\|^2_F + \frac{\lambda}{2}(\text{tr} \epsilon)^2$

• if you solve for $\mu$ and $\lambda$ to provide the same energy when $\epsilon_l = -\nu \epsilon_l$, you get the formulas:

  . $\mu = \frac{E}{2(1 + \nu)}$ and $\lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}$ (in 3D) or $\lambda = \frac{E\nu}{(1 + \nu)(1 - \nu)}$ (in 2D)
Linear strain

For small deformations we can use a first-order approximation to $E$

- $E(F) \approx E(I) + \delta E(I)|_{\delta F = F - I} + \ldots$

  $E(F) \approx \delta E(I)|_{\delta F = F - I}$
  $= \frac{1}{2} \left( (F - I)^T + (F - I) \right)$

- $= \frac{1}{2} (F + F^T) - I$

$\delta E(F) = \frac{1}{2} \delta [F^T F - I]
= \frac{1}{2} \left( \delta [F]^T F + F^T \delta [F] \right)$

$\delta E(I) \approx \frac{1}{2} \left( \delta [F]^T I + I^T \delta [F] \right)
= \frac{1}{2} \left( \delta [F]^T + \delta [F] \right)$

infinitesimal (linear) strain: $\epsilon = \frac{1}{2} \left( F + F^T \right) - I$
Linear elasticity

The first constitutive model for an isotropic material

- measure deformation using the linear strain $\epsilon = \frac{1}{2} (F + F^T) - I$
- define strain energy density as $\psi = \mu \| \epsilon \|^2_F + \frac{\lambda}{2} (\text{tr} \, \epsilon)^2$

To determine forces on mesh vertices we need $\partial \psi / \partial x_i$

- the chain-rule chain is $x \rightarrow F \rightarrow \epsilon \rightarrow \psi \rightarrow E$
- we already derived $\partial F / \partial x$ and $\partial E / \partial \psi$ will be simple
- we still need $\partial \epsilon / \partial F$ and $\partial \psi / \partial \epsilon$ (these are the ones that depend on the material model)
- will work these two out using variational notation and derive a formula for

$$P(F) = \frac{\partial \psi(F)}{\partial F}$$

known as the “first Piola-Kirchhoff stress”
Energy density gradient for linear elasticity

First the derivative of elastic energy density with respect to strain

\[ \delta \psi(F) = \delta [\mu \| \epsilon(F) \|_F^2 + \frac{\lambda}{2} (\text{tr} \, \epsilon(F))^2] \]

\[ = 2\mu \epsilon : \delta \epsilon + \lambda (\text{tr} \, \epsilon) I : \delta \epsilon \]

next the derivative of strain with respect to deformation gradient

\[ \epsilon = \frac{1}{2} (F + F^T) - I \]

\[ \delta \epsilon = \delta [\text{sym } F] = \text{sym } \delta F \quad \text{where} \quad \text{sym } A = \frac{1}{2} (A + A^T) \]

substituting:

\[ \delta \psi(F) = (2\mu \epsilon + \lambda (\text{tr} \, \epsilon) I) : \delta F \quad \text{—simplified via } S : \text{sym } A = S : A \text{ for symmetric } S \]

\[ P = 2\mu \epsilon + \lambda (\text{tr} \, \epsilon) I \quad \text{—this is } \partial \psi / \partial F \]
Computing nodal forces

Now we have the complete chain of derivatives for the first model

- let’s compute the forces as $f_i = - \partial E/\partial x_i$

  $$E[\phi] = \int_B \psi(F(X)) \, dX$$

  $$E[\phi] = \sum_k \int_{T_k} \psi(F(X)) \, dX = \sum_k |T_k| \psi(F_k)$$

- recall that $F = DD_0^{-1}$, so $\delta F = \delta[D] D_0^{-1}$

- we have $\delta \psi = P : \delta F$, so $\delta \psi = P : (\delta[D] D_0^{-1}) = PD_0^{-T} : \delta D$

- for triangle $k$, $\delta E = |T_k| \delta \psi = |T_k| PD_0^{-T} : \delta D$

- thus $\delta E = - H : \delta D$ where $H = - |T_k| PD_0^{-T}$

- $H = \begin{bmatrix} f_1 & f_2 \end{bmatrix}$ and $f_0 = -(f_1 + f_2)$
To sum up...

So you are writing a simulator and need to compute forces on your vertices?

No problem, follow these 5 steps:

1. Ahead of time compute $D_0^{-1}$ and $|T_k|$.

2. Compute $F = DD_0^{-1}$ from the current vertex positions.

3. Compute the strain from $F$ using the formulas appropriate to your model.

4. Compute the stress $P$ from the strain using the formulas appropriate to your model.

5. Compute $H = |T_k|PD_0^{-T}$, and the forces are sitting in the columns of $H$. 

$D_0^{-1}$ 
$T_k$ 
$F = DD_0^{-1}$ 
$P$ 
$H = |T_k|PD_0^{-T}$