05 Mass-and-spring models
Binary Spring

The most basic modeling tool for all kinds of deformable things

Spring defined by
• which particles $i$ and $j$ it connects
• its spring stiffness $k_s$
• its rest length $l_0$

From Hooke’s law we know force is proportional to displacement from rest
• $f = k_s(l - l_0)$

The force acts along the direction of the spring
• $f_i = k_s(\|x_{ij}\| - l_0)\hat{x}_{ij}$ where $x_{ij} = x_j - x_i$ (quick sanity check: pulls $i$ towards $j$ when stretched)
• $f_j = k_s(\|x_{ji}\| - l_0)\hat{x}_{ji} = -f_{ij}$
Adding damping

Springs are usually too springy!
- a system of springs will oscillate forever…

Damping will dissipate energy
- but using the drag forces $\mathbf{f}_d = -k_d \mathbf{v}$ we use for free particles can slow things inappropriately
- this force opposes all motion; we only want to oppose the spring’s movement

Spring damping force opposes changes in spring length only
- only opposes relative motion
- only opposes motion in the direction of the spring
- $\mathbf{f}_i = k_d (\mathbf{v}_{ij} \cdot \hat{\mathbf{x}}_{ij}) \hat{\mathbf{x}}_{ij}$ where $\mathbf{v}_{ij} = \mathbf{v}_j - \mathbf{v}_i$ (sanity check: pulls $i$ towards $j$ when elongating)
Phenomena of damped springs

Oscillations of undamped springs

- a 1D damped spring obeys \( m\ddot{x} + k_d\dot{x} + k_s x = 0 \)

- when \( k_d \) is negligible the solution is \( x(t) = C_1 \cos(\omega t + C_2) \) where \( \omega^2 = \frac{k_s}{m} \)
  - stiffer spring \( \rightarrow \) faster oscillation; higher mass \( \rightarrow \) slower oscillation

- when \( k_s \) is negligible the solution is \( x(t) = c_0 + c_1 T(1 - e^{t/T}) \) where \( T = \frac{m}{k_d} \)
  - higher mass \( \rightarrow \) takes longer to stop; higher damping \( \rightarrow \) takes less time to stop

- in general case we get \( x(t) = Ce^{-t/T} \cos(\omega t + C_2) \) where
  \[
  T = \frac{2m}{k_d} \quad \text{and} \quad \omega = \sqrt{\frac{k_s}{m} - (1/T)^2}
  \]
  - decaying oscillation combining the two above behaviors
  - damping also slows the oscillation; when we reach \( \omega = 0 \) the system is “critically damped”
A rod is a long, slender, flexible object (essentially 1D)
  • can stretch or bend and elastically resist both

Basic plan: a chain of masses and springs
  • $N$ springs of length $L/N$ and spring constant ... $Nk_s$ (why?)
  • this handles stretching but doesn’t oppose bending—too “floppy” and chain-like

Simple way to resist bending: bending springs
  • springs that skip one particle
  • $N - 1$ springs of length $2L/N$ and spring constant $Nk_s'$ where $k_s'$ is usually quite a bit less than $k_s$
Modeling cloth with springs

**Very similar idea to rods, but in a 2D grid**

- structural springs along the axes (stiff for woven cloth, softer for knitted)
- bending springs skipping one particle (weak to allow lots of bending)
- shear springs along the diagonals (weaker than structural, to allow shearing)

**You’ll do this in the assignment!**

- it’s only cloth if it’s in 3D, so can’t really demo this ;)

![Diagram of cloth simulation with springs](image)
index particles in 2D

• avoids row-column indexing calculations

index springs the same way

• think of all springs connected to one particle

• compute forces by visiting all particles and considering all springs connected to that particle

• (to enumerate the springs, visit particles and considering only the springs connected to later particles)
Modeling deformable solids with springs

2D: looks a lot like cloth
  • don’t need bending springs
  • shear springs should probably be stronger than for cloth
  • a triangular mesh only requires springs along the edges
  • in a 2D space, a 2D object can resist compression

3D: need enough springs to prevent collapsing
  • for a cube mesh, various strategies are possible — bracing diagonals of faces, or bracing across the diagonals of the cube
  • a tetrahedral mesh is naturally stable with just a spring along each edge
Binary springs are simple and are a lot of fun to play with but they eventually start to become limited

- bending and shear springs contribute also to stretching stiffness
- difficult to achieve behavior matching particular measurements or material models
- bending springs are not very good at resisting slight bending (bending stiffness = 0 when straight!)
- difficult or impossible to express things like volume or area preservation

The spring force belongs to a useful class

- it is a conservative force, meaning it takes the same amount of work to get from one configuration to another regardless of the path
- this means it is the derivative (gradient in this case) of a potential … and the potential is literally the potential energy stored in the spring!
Working out our spring force from the energy

Start with the spring energy

\[ E_{ij}(\mathbf{x}) = \frac{1}{2} k_s (\|\mathbf{x}_i - \mathbf{x}_j\| - l_0)^2 \] (this is the contribution of one spring to the total system energy)

Force is minus the gradient of energy

\[ f_i(\mathbf{x}) = \frac{\partial E}{\partial \mathbf{x}_i}(\mathbf{x}) \] (remember \(\mathbf{x}\) is a big vector of all the positions; this partial derivative is zero for all the particles that are not connected to this particular spring)

Take the computation one step at a time:

- derivative of \(\mathbf{x}_i - \mathbf{x}_j\)
- derivative of \(\|\mathbf{v}\|\) wrt. \(\mathbf{v}\)
- derivative of \(E_{ij}\) wrt \(\|\mathbf{v}\|\)