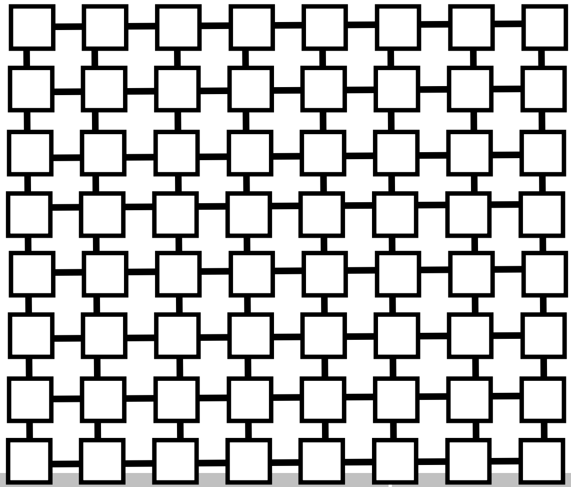
s-t Graph Cuts for Binary Energy Minimization

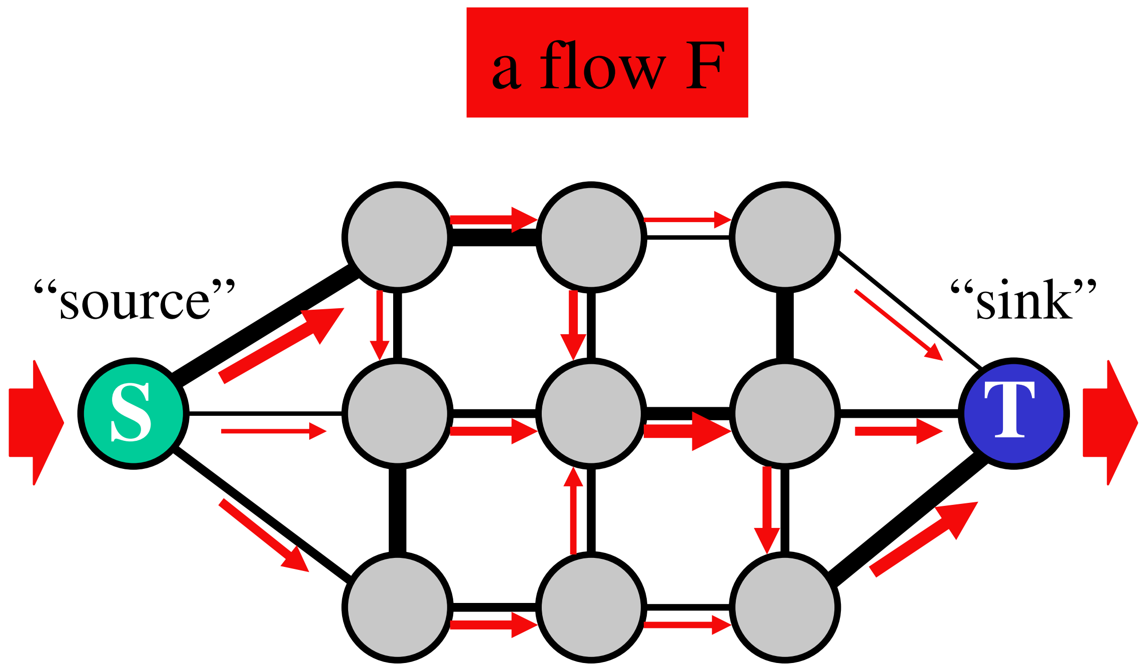
$$E(L) = \underbrace{\sum_p D_p(L_p)}_{\text{data term}} + \underbrace{\lambda \sum_{pq \in N} I(L_p \neq L_q)}_{\text{prior term}}$$

- Now that we have an energy function, the big question is how do we minimize it?



- Exhaustive search is exponential: if n is the number of pixels, there are 2^n possible labelings L

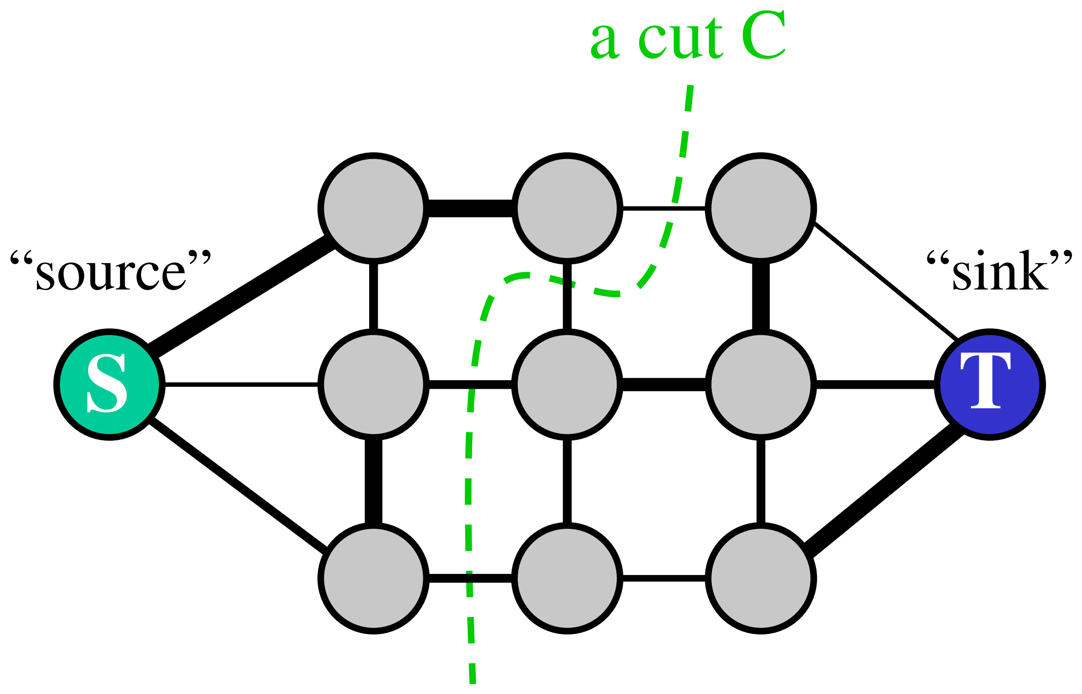
Maximum flow problem



A graph with two terminals

- Max flow problem:
  - Each edge is a "pipe"
  - Find the largest flow F of "water" that can be sent from the "source" to the "sink" along the pipes
  - Source output = sink input = flow value
  - Edge weights give the pipe's capacity

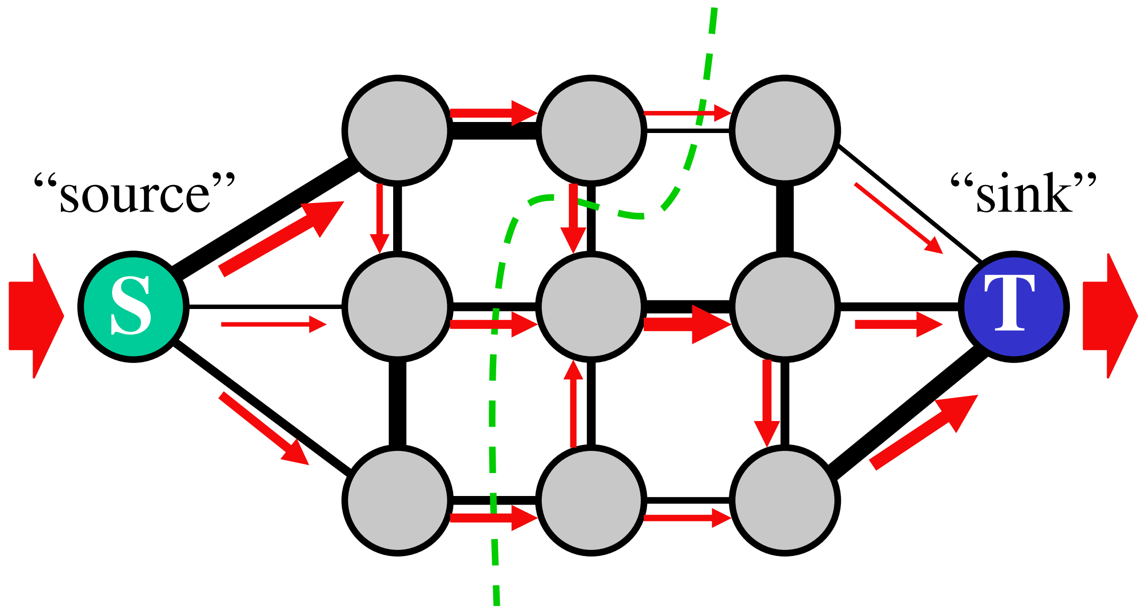
Minimum cut problem



A graph with two terminals

- Min cut problem:
  - Find the cheapest way to cut the edges so that the "source" is separated from the "sink"
  - Cut edges going from source side to sink side
  - Edge weights now represent cutting "costs"

Max flow/Min cut theorem



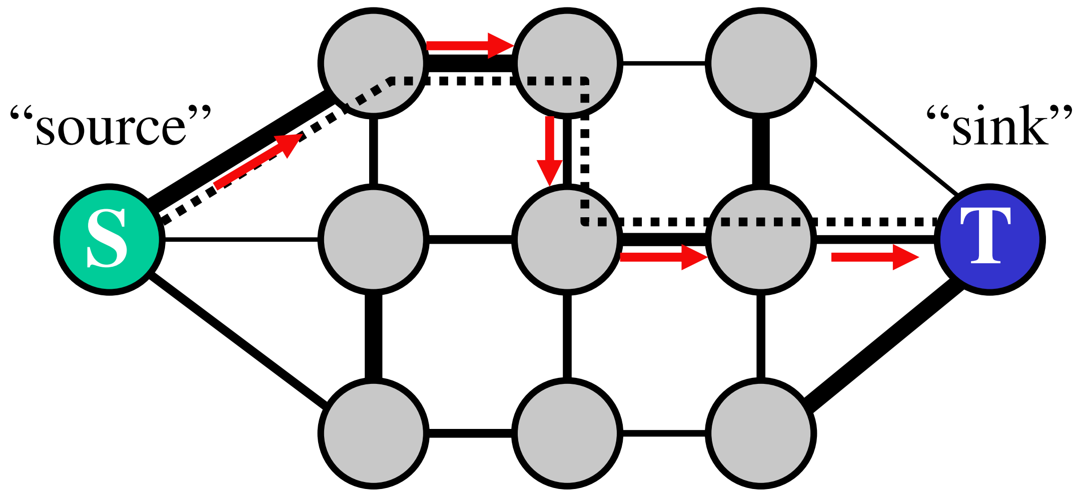
A graph with two terminals

- Max Flow = Min Cut:
  - Proof sketch: value of a flow is value over any cut
  - Maximum flow saturates the edges along the minimum cut
    - Ford and Fulkerson, 1962
    - Problem reduction!
- Ford and Fulkerson gave first polynomial time algorithm for globally optimal solution

Fast algorithms for min cut

- Max flow problem can be solved fast
  - Many algorithms, we'll sketch one
- This is not at all obvious
  - Variants of min cut are NP-hard
- Multiway cut problem
  - More than 2 terminals
  - Find lowest cost edges separating them all

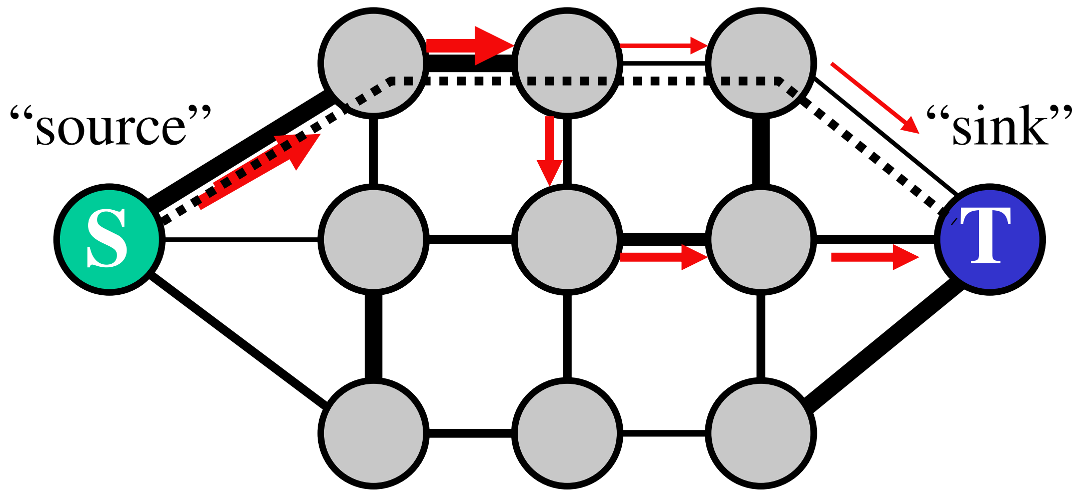
"Augmenting Path" algorithms



A graph with two terminals

- Find a path from S to T along non-saturated edges
- Increase flow along this path until some edge saturates

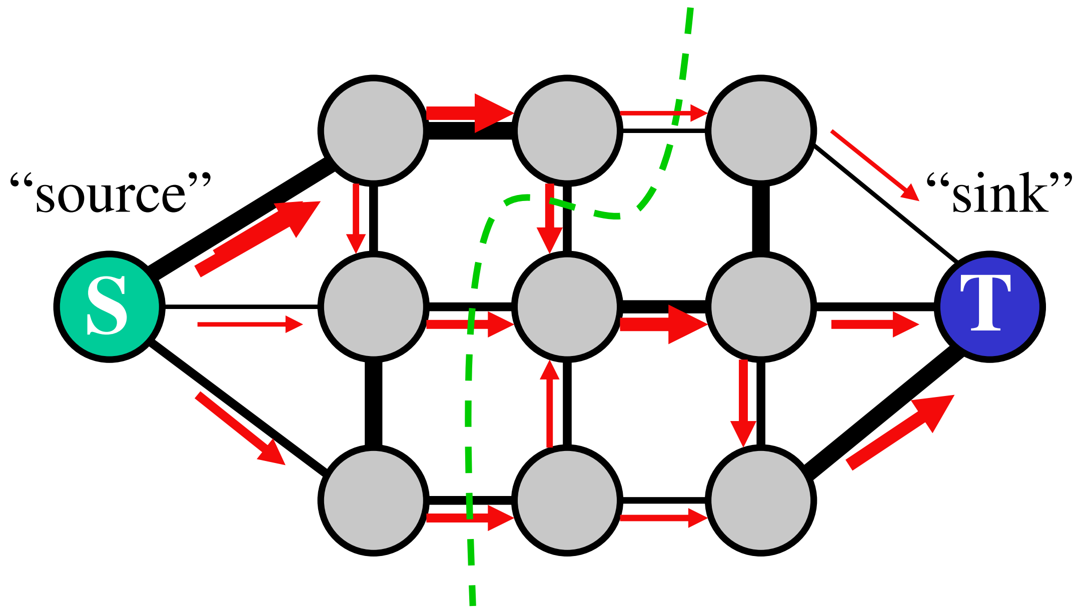
"Augmenting Path" algorithms



A graph with two terminals

- Find a path from S to T along non-saturated edges
- Increase flow along this path until some edge saturates
- Find next path…
- Increase flow…

"Augmenting Path" algorithms



A graph with two terminals

- Find a path from S to T along non-saturated edges
- Increase flow along this path until some edge saturates

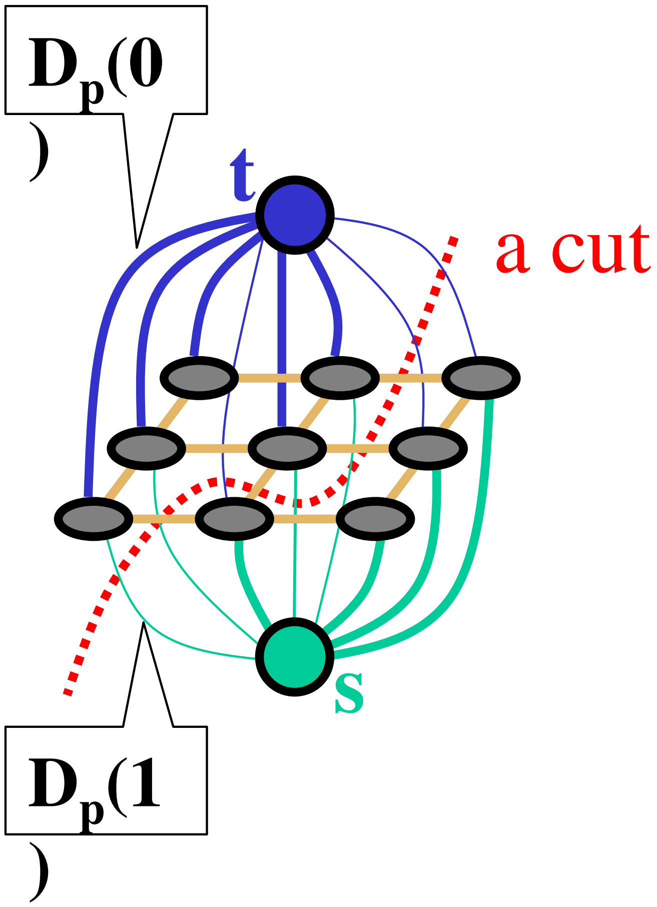
Iterate until …
all paths from S to T have at least one saturated edge

Implementation notes

- There are many fast flow algorithms
- Augmenting paths depends on ordering
  - Breadth first = Edmonds-Karp
  - Vision problems have many short paths
  - Subtleties needed due to directed edges
- [BK ’04] gives an algorithm especially for vision problems
  - Software is freely available

Basic construction

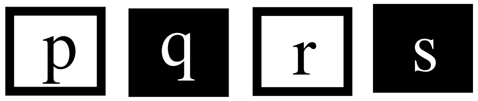
- One non-terminal vertex per pixel
- Each pixel connects directly to s,t
  - Severing these edges corresponds to giving labels 0,1 to the pixel
- Cost of cut is the cost of the entire labeling



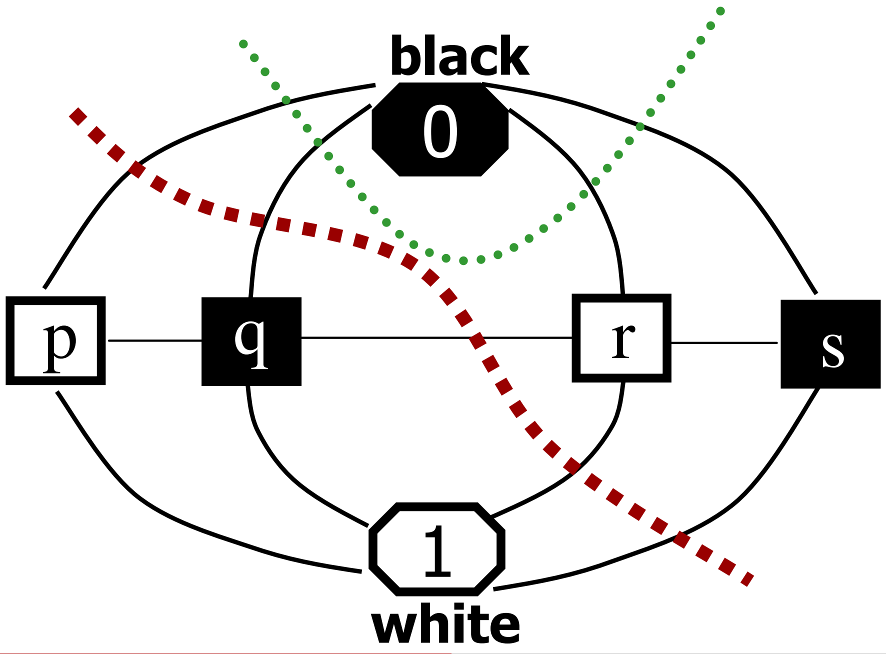
Example

$$E(L) = \sum_{p} D_p(L_p) + \lambda \sum_{pq \in N} I(L_p \neq L_q)$$

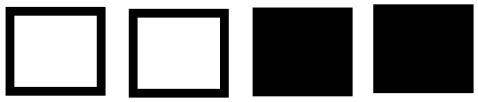
- For clarity, let's look at 1 dimensional example but everything works in 2 or higher dimensions. Suppose our image has 4 pixels:



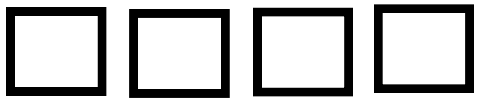
- We build a graph:



- The cut in red corresponds to labeling



- The cut in green corresponds to labeling



Beyond binary faxes

- All we really need is a cost function
  - Suppose that label 1 means "foreground" and label 0 means "background"
- How do we figure out if a pixel prefers to be in the foreground or background?
  - Predefined intensities: for instance, the foreground object tends to have intensities in the range 50-75
    - So if we observed an intensity in this range at p, $D_p(1)$ is small
  - This is **not** image thresholding!

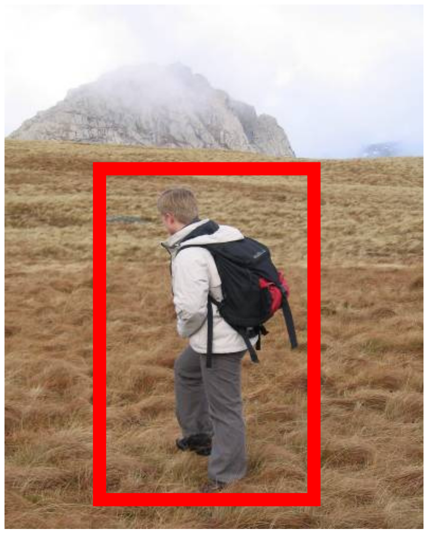
Better intensity models

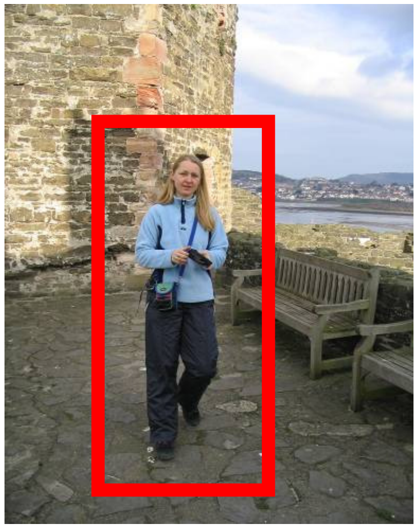
- We can compute the range of intensities dynamically rather than statically
  - Both for foreground and for background
- User marks some pixels as being foreground, and some as background
  - Compute a D_p based on this
  - For instance, $D_p(1)$ is small if p looks like pixels marked as being foreground
- Based on the resulting segmentation, mark additional pixels

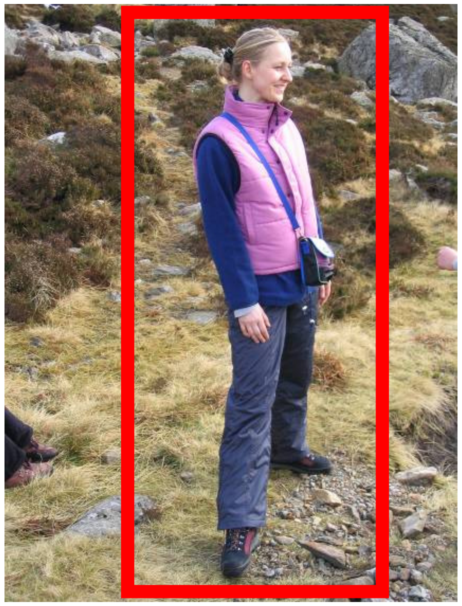
A serious implementation

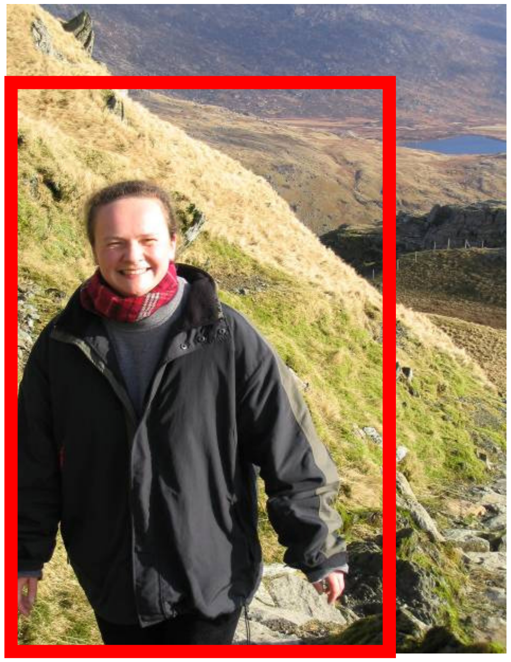
- This basic segmentation algorithm is now in a Microsoft product
  - SIGGRAPH paper on “GrabCut”
- Same basic idea, but simpler UI
  - Assume the background is outside and the foreground is inside a user-supplied box
  - $D_p(1)$ is small if p looks like the pixels inside the box, large if like the pixels outside
  - Create a new segmentation and use it to re-estimate foreground and background

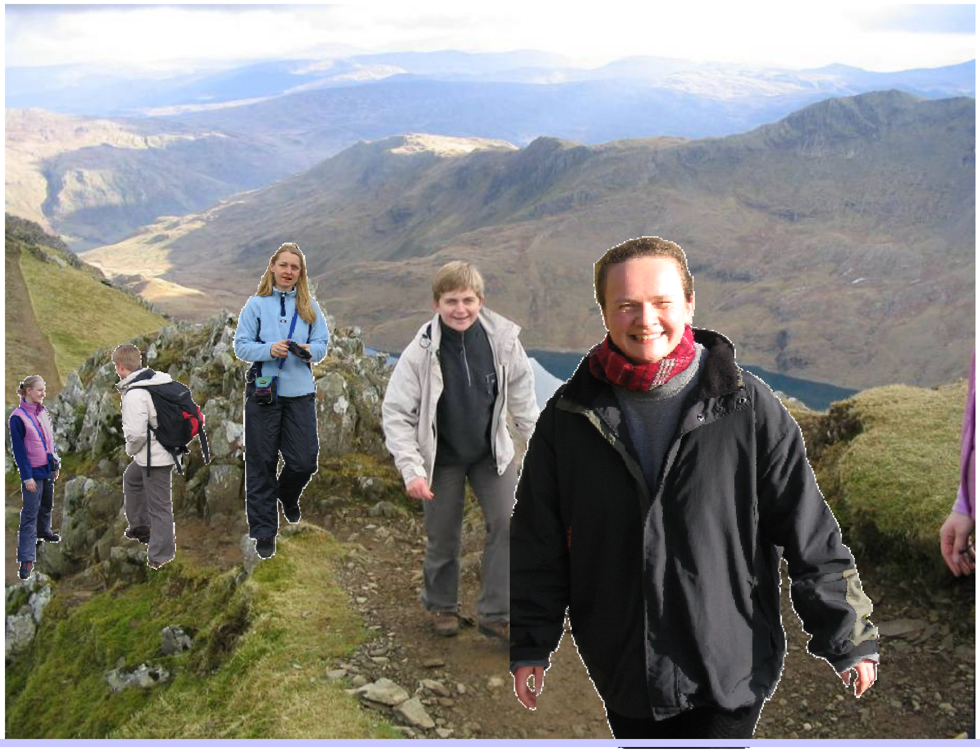
GrabCut





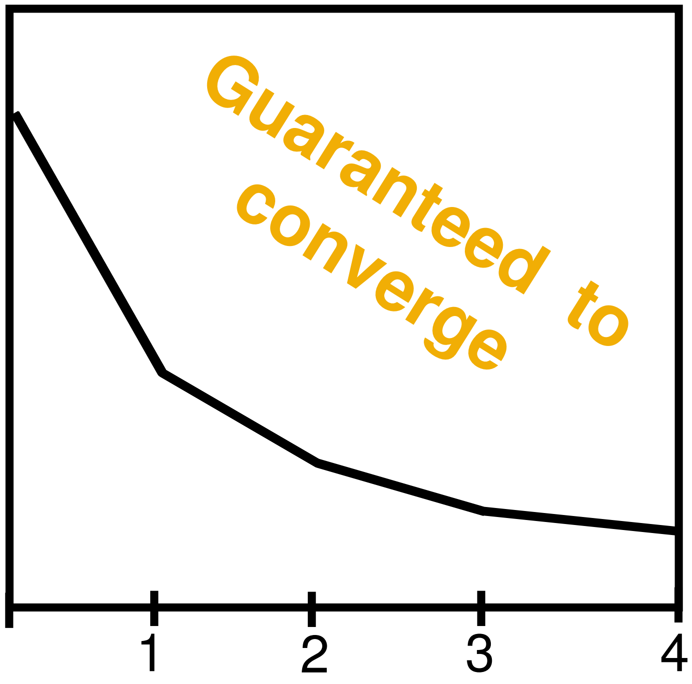






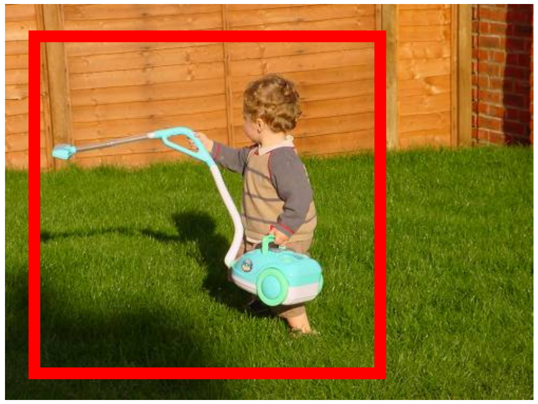
Iterated Graph Cuts

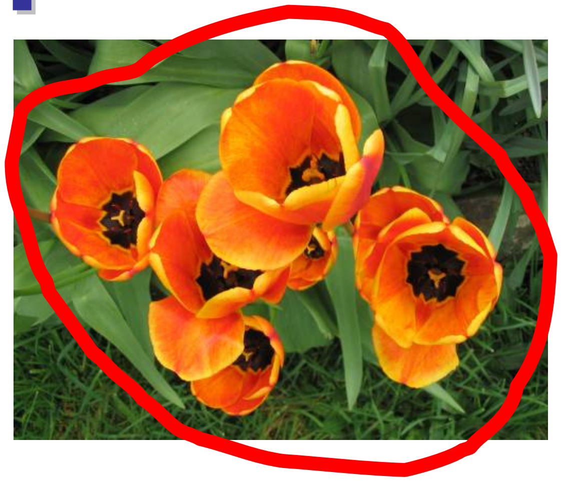
Result

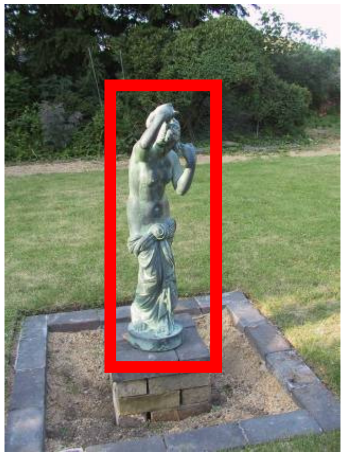


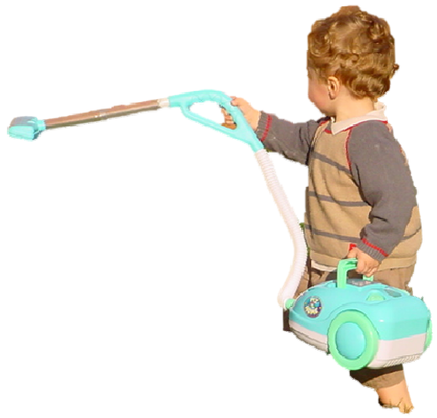
See: http://www.youtube.com/watch?v=9jNB6fza0nA&feature=related

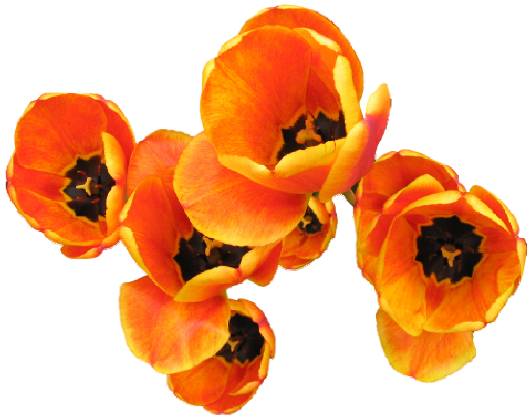
Moderately straightforward examples

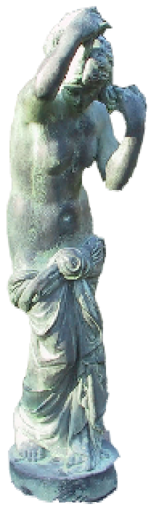












... GrabCut completes automatically

Important properties

- Very efficient in practice
  - Lots of short paths, so roughly linear
- Construction is symmetric (0 vs 1)
- Specific to 2 labels
  - Min cut with >2 labels is NP-hard

Can this be generalized?

- NP-hard for Potts model [K/BVZ 01]
- Two main approaches

1. Exact solution [Ishikawa 03]
   - Large graph, convex V (arbitrary D)
   - Not the considered the right prior for vision
2. Approximate solutions [BVZ 01]
   - Solve a binary labeling problem, repeatedly
   - Expansion move algorithm