1

We observe that once the stock price hits the upper level $S_u$, the barrier option from the problem becomes a down-and-in option and once the lower level $S_d$ is hit, then the derivative becomes worthless.

For $S_d < S < S_u$ and $0 < t < T$, the value $V(S,t)$ of the barrier option (before any price level is reached) should satisfy the BS equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S \frac{\partial V}{\partial S} - rV = 0.$$ 

The final condition is:

$$V(S,T) = 0, \quad S_d \leq S \leq S_u.$$ 

The two boundary conditions are:

$$V(S_d, t) = 0, \quad 0 \leq t \leq T,$$
$$V(S_u, t) = V_{\text{down-and-in}}(S_u, t), \quad 0 \leq t \leq T,$$

where $V_{\text{down-and-in}}(S, t)$ is the value of the knock-in option that solves also the BS equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S \frac{\partial V}{\partial S} - rV = 0, \quad S_d < S < \infty, \quad 0 < t < T$$

with the final condition

$$V_{\text{down-and-in}}(S,T) = 0, \quad S_d \leq S \leq S_u$$

and the boundary conditions:

$$V_{\text{down-and-in}}(S_d, t) = 1,$$
$$V_{\text{down-and-in}}(S, t) = 0, \quad S \to \infty.$$
(a) function avg=AveragePath(S0,T,r,sig,tw,N,M)
dt=tw/M;
m=zeros(N,1);
a=rand(M+1,N);
S=zeros(M+1,N);
S(1,:) = S0*exp((r-sig^2/2)*(T-tw)+sig*sqrt(T-tw)*a(1,:));
for i = 2:M+1
    S(i,:) = S(i-1,:)*exp((r-sig^2/2)*dt+sig*sqrt(dt)*a(i,:));
end
avg=mean(S);

(b) function V= EuroAvgRateCall(S0,T,E,r,sig,tw,N,M)
avg = AveragePath(S0,T,r,sig,tw,N,M);
V0 = exp(-r*T)*max(avg-E,0);
V = mean(V0);

tw = 3/12;
r = .05;
M = 30;
N = 20000;
S0 = 100;
T = 1;
sig = .3;
E = 100;
n = 500;
V = zeros(n,1);
for i = 1:n
    V(i) = EuroAvgRateCall(S0,T,E,r,sig,tw,N,M);
end
V0 = mean(V)
standdev = std(V)

(c) The only function we have to change in order to include the antithetic technique is AveragePath:

function avg = AveragePath(S0,T,r,sig,tw,N,M)

dt=tw/M;
q=1:(M+1);
m=zeros(N,1);
b=rand(M+1,N/2);
a=[b -b]' t = T-tw+(q-1)*dt;
S=zeros(M+1,N);
S(1,:) = S0*exp((r-sig^2/2)*(T-tw)+sig*sqrt(T-tw)*a(1,:));
for i = 2:M+1
    S(i,:) = S(i-1,:)*exp((r-sig^2/2)*dt+sig*sqrt(dt)*a(i,:));
end
avg=mean(S);
The average (the price of the option) in the first case is 12.8456 and for the antithetic technique is 12.8436. The standard deviations are 0.1400 and 0.1080 respectively. The antithetic methods will give a faster convergence.

3

(a) When we discretize using the Euler method the following equation:

\[ dr_t = \kappa (\mu - r_t) dt + \sqrt{\sigma r_t} dX_t \]

we get:

\[ r_{t+1} = r_t + \kappa (\mu - r_t) dt + \epsilon_t \sqrt{\sigma r_t} dt, \]

where \( \epsilon_t \) is a normally distributed variable.

(b) function \([\text{avgrate}, rT] = \text{CIRPath}(r0, T, k, \mu, \sigma, N, M)\)

\[
\begin{align*}
\text{r} &= \text{zeros}(M+1,N); \\
r(1,:) &= r0; \\
\text{dt} &= T/M; \\
\text{for } k &= 2:(M+1) \\
\text{a} &= \text{randn}(1,N); \\
r(k,:) &= r(k-1,:) + k^* (\mu - r(k-1,:))^* dt + \text{sqrt}((\sigma^* r(k-1,:))^* dt)^* a; \\
\text{end} \\
\text{avgrate} &= \text{mean}(r,1); \\
rT &= r(M+1,:); \\
\end{align*}
\]

(i)–(ii) M = 200; \\
N = 40000; \\
T = 1; \\
r0 = 0.05; \\
k = 0.2; \\
\mu = 0.08; \\
\sigma = 0.05; \\
[\text{avgrate}, rT] = \text{CIRPath}(r0, T, k, \mu, \sigma, N, M); \\
\text{E1} = \text{zeros}(N,1); \\
\text{E1} &= \text{exp}(-\text{avgrate})^* \text{max}(rT-.06,0); \\
\text{E} &= \text{mean}(\text{E1}); \\
\text{E2} &= \text{exp}(-\text{avgrate}); \\
\text{e2} &= \text{mean}(\text{E2}); \\
\text{E3} &= \text{max}(rT-.06,0); \\
\text{e3} &= \text{mean}(\text{E3}); \\
e &= e2^* e3;
\]

The results we get are in the range \(8.1502 \cdot 10^{-4} \) and \(8.2816 \cdot 10^{-4}\) respectively. We observe that the results of the two methods are close, but they are not exactly the same. This is because the expectation of the product of two functions is not equal to the product of the expectations of the two functions. This would become better noticed if we would take \( T \neq 1 \). Among these methods, the better one is the first one.
(c) We cannot use the results calculated above for the bond prices because the expectations are calculated under the empirical probabilities and the bond prices should be calculated as expectations under the risk neutral probabilities. So if we denote by \( Q \) the risk neutral probabilities, then the price of a bond at time 0 with maturity \( T \) is given by:

\[
E_Q(e^{-\int_0^T r_s ds}).
\]

(d) Given the arbitrage-free bond expectations and assuming that the short interest rate follows the Cox-Ingersoll-Ross model, in order to find the arbitrage-free dynamics of \( r_t \) we need to find the market price of risk \( \lambda \). The estimated price of interest rate risk is different from zero. However, this does not necessarily translate into a significant difference in pricing interest rate dependent assets. The price of a zero coupon bond with payoff \( \$1 \) at time \( T \) as we said above is given by:

\[
P(0, T) = E_Q(e^{-\int_0^T r_s ds}).
\]

Also, the spot rate satisfies under the risk neutral probabilities:

\[
dr_t = [\mu(r_t) - \lambda(r_t)\sigma(r_t)]dt + \sigma(r_t)dZ_t,
\]

where \( Z_t \) is the Brownian motion under the new risk-adjusted probability.

The expectation in the equation of the bond price can be calculated using Monte Carlo simulation using different paths of \( r_t \) but under measure \( Q \). Incorporating the estimated market price of risk makes a significant difference to the results, the size of the difference increasing as the maturity of the bond increases. So this is why the estimations from the first part of the problem were not correct. Since zero-coupon bond prices differ a lot, we can expect even greater differences for more complex interest rate derivatives.

In the case of the Cox-Ingersoll-Ross model the market price of risk, unobservable value, is \( \lambda(r_t) = \lambda \sqrt{T} \).