1. (20 points) Assume that there is no dividend. Prove the following inequality for an American call option:

\[ C \geq S - E e^{-r(T-t)}. \]

Show that the holder of the American call option maximizes its value by holding the option to expiration.

2. (30 points) Assume that an underlying stock price is modeled as a geometric Brownian motion with a constant volatility \( \sigma > 0 \). Assume that the risk-free interest rate \( r > 0 \) is a constant. To price an option on this underlying, an \( M \)-step binomial tree can be constructed, e.g., \( \delta t = \frac{T}{M} \), \( t_j = j \delta t, \ j = 0, \ldots, M \), using for example the Cox-Ross-Rubinstein (CRR) model, \( u = e^{\sigma \sqrt{\delta t}} \) and \( d = \frac{1}{u} \). Assume now that the stock goes ex-dividend at a discrete set of times \( 0 < \tilde{t}_1 < \cdots < \tilde{t}_k \leq T \) with a dividend yield \( q > 0 \). The CRR binomial model can be extended for discrete dividend payments as follows: let \( S_0^0 = S_0 \), and for each \( t_j, \ 1 \leq j \leq M \), let \( \tilde{t}_i \leq t_j < \tilde{t}_{i+1} \) (here we can define \( \tilde{t}_0 = 0 \) for notational simplicity),

\[ S_i^j = u^j d^{j-i}(1-q)^j S_0^0, \ i = 0, 1, 2, \ldots, j. \]

(a) Assume that the stock goes ex-dividend at \( t + 1 \) and consider a 1-period binomial model: at \( t + 1 \), \( S_{i+1} \) is either \( u(1-q)S_i \) or \( d(1-q)S_i \) with \( u > d > 0 \). Assume that \( t + 1 = T \) for simplicity. Show that the time \( t \) value \( V_i \) of the Bermuda option which can be exercised at \( t \) and \( T = t + 1 \) satisfies

\[ V_i = \max \left( e^{-r\delta t} \left( p_i V_{i+1}^u + (1-p_i) V_{i+1}^d \right), \max(E - S_i, 0) \right) \]

where \( V_{i+1}^u, V_{i+1}^d \) denote the option values at \( t + 1 \) when stock price goes up and down respectively, and \( p^i \) is the risk neutral probability

\[ p^i = \frac{e^{r\delta t} - d}{u - d}. \]

(b) Using the above formula, write a Matlab function

\[ V_0 = \text{MyBinBermuda}(S_0, E, T, r, \sigma, M, \tilde{t}, q, \text{flag}) \]

to compute the initial Bermuda call/put option value with a strike \( E > 0 \) and expiry \( T > 0 \); the Bermuda option is only exercisable at ex-dividend dates \( \tilde{t} \) and
the expiry $T$. Here $\hat{t}$ is a $k$-vector of ex-dividend dates $\hat{t}_1 < \cdots < \hat{t}_k$; no dividend is paid if $\hat{t}$ is set to empty, i.e., $\hat{t} = \emptyset$. If an exercise time $\hat{t}_l$, $l = 1, \cdots, k$, does not equal to any $t_j$ and assume $t_{j-1} < \hat{t}_l < t_j$, choose $t_j$ as an exercisable time as an approximation.

Your **MyBinBermuda** returns the initial Bermuda call price when flag = 1 and the Bermuda put price when flag = 0.

(c) Let $S_0 = 100$, $r = 0.05$, $q = 0.02$, $\sigma = 0.2$, $M = 100$, and $T = 1$. In one plot, graph the function $\max(E - S_0, 0)$ and the initial Bermuda put option prices computed using **MyBinBermuda** against the strike $E = 50 : 1 : 150$ assuming ex-dividend dates of $\hat{t} = [1/12, 2/12, \cdots, 1]$. What is the the observed relationship between the payoff and the option value? Explain.

3. (50 points) Assume that the underlying price satisfies the following stochastic differential equation

$$
\frac{dS_t}{S_t} = (\mu - q)dt + \sigma dX_t,
$$

where the expected rate of return $\mu$ and the continuous dividend yield $q$ are positive constants, $X_t$ is a standard Brownian motion, and $\sigma > 0$ is the constant volatility. Assume that the interest rate $r > 0$ is a constant. The American option value $V(S, t)$ solves the following parabolic partial differential complementarity problem,

$$
\begin{cases}
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S \frac{\partial V}{\partial S} - r V & \leq 0 \\
V(S, t) - \text{payoff}(S) & \geq 0 \\
(V(S, t) - \text{payoff}(S)) \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S \frac{\partial V}{\partial S} - r V\right) = 0
\end{cases}
$$

with the final condition $V(S, T) = \text{payoff}(S)$. Consider the same discretization as in Assignment 3: assume that $S_{\max} > 0$ is sufficiently large and $T$ is the expiry of the option. Let $\delta t = \frac{T}{M}$, $\delta S = \frac{S_{\max}}{N}$, $t_j = (j - 1) \cdot \delta t$, and $S_i = (i - 1) \cdot \delta S$. Let $\{(S_i, t_j)\}_{i=1, \cdots, M+1, j=1, \cdots, M+1}$ denote a uniform discretization of the region $[0, S_{\max}] \times [0, T]$. Let $V^j_i$ denote the option value at $t_j$ and $S_i$. You are to implement the **implicit** finite difference method for American option.

(a) What are the appropriate boundary conditions (at $S = 0$ and $S \to +\infty$) for American call and put respectively?

(b) Let $x = [V^{j-1}_2; \cdots; V^{j-1}_N]$ and $g = [\text{payoff}(S_2); \cdots; \text{payoff}(S_N)]$. The finite difference approximation of the partial differential complementarity problem at time $t_{j-1}$, using the **implicit method**, can be written as

$$
Ax - b^j \geq 0, \quad x \geq g \\
(x - g) \cdot (Ax - b^j) = 0
$$

i. Write down the coefficient matrix $A$ and $b^j$.

ii. The following procedure is often used to approximate American option values:
Compute $V_i^{M+1}, i = 1, \cdots, N + 1$

for $j = M + 1 : -1 : 2$

Compute $\{V_i^{j-1}, V_i^{j+1}\}$

solve $Ay = b^j$

$x_i = \max(y_i, \text{payoff}(S_i)), i = 1, \cdots, N - 1$

$V_i^{j-1} = x_i, i = 1, \cdots, N - 1$

end

For each $j$, does the computed $x$ from the above solve the corresponding linear complementarity problem (2) from implicit finite difference method? Explain.

iii. Using the implicit finite difference, TriDiLU, LBidiSol, and UBidiSol, write a Matlab function,

$$V = \text{PDEAmeri}(r, q, \sigma, E, T, M, N, S_{\text{max}}, \text{flag})$$

which returns a $(N + 1)$-vector of the initial price of the American option with the strike $E > 0$ and expiry $T > 0$; the $i$-th component $V_i$ equals the computed option premium at $S_i = (i - 1)\delta S$ (it is assumed here that there is a single strike and expiry). The initial call price is returned when flag equals 1 and put price is returned when flag equals zero.

(c) Write the following Matlab function

$$function \ x = \text{PSOR}(x, e, d, f, g, b, \omega, \text{tol}, \text{imax})$$

to solve a complementarity problem

$$Ax - b \geq 0, x \geq g$$

$$(x - g) \cdot (Ax - b) = 0$$

by the projected SOR method. Here the input arguments $1 < \omega < 2$, and tol, and imax are over-relaxation parameter and stopping tolerance, and maximum number iterations allowed in PSOR, respectively.

(d) Using implicit finite difference method, write a Matlab function

$$V = \text{AmeriSOR}(r, q, \sigma, E, T, M, N, S_{\text{max}}, \text{flag}, \omega, \text{tol}, \text{imax})$$

which returns a $(N + 1)$-vector of the initial American option price with the strike price $E > 0$ and expiry $T > 0$. The function returns the call prices when flag equals 1 and put prices when flag equals zero.

(e) Let $S_0 = 100, r = 0.05, q = 0.02, \sigma = 0.2, M = 100, N = 200, E = 90, T = 1$, and $S_{\text{max}} = 2S_0$. Let $\text{tol} = \sqrt{\epsilon}$ where $\epsilon$ is the machine epsilon in Matlab and determine the over-relaxation $\omega$ by trial and error. Set a suitable value for imax.
i. In one plot, graph the put payoff function and the computed initial put prices, using **AmeriSOR**, against the underlying asset price. In a separate plot, graph the difference of the computed prices using **AmeriSOR** and **PDEAmeri** against the underlying price.

ii. Using central finite differences, compute the delta hedge parameter

$$\Delta_i \approx \frac{V_{i+1} - V_{i-1}}{2(\delta S)} , \; i = 1, \cdots N$$

Graph the delta values $\Delta$ computed using **AmeriSOR** against the corresponding asset price $S$. In a separate plot, graph the difference of the computed delta using **AmeriSOR** and **PDEAmeri** against the underlying price.