1. (75 points) Assume that the stock price follows the binomial model described in Assignment 1:

\[ S_{t+1} - S_t = \mu \cdot S_t \cdot \delta + \sigma \cdot S_t \cdot \sqrt{\delta} \cdot \epsilon_t \]  

(1)

where \( \epsilon_t \) is a serially uncorrelated binomial process assuming the following values

\[ \epsilon_t = \begin{cases} 
+1 & \text{with probability } \frac{1}{2} \\
-1 & \text{with probability } \frac{1}{2} 
\end{cases} \]

\( t = 0, 1, \ldots, M \), and \( \delta = \frac{T}{M} \) is the length of the time interval; \( S_t \) denotes the stock price at time \( t_i = i \cdot \delta \). Assume that the continuously compounded interest rate \( r \) is 5%.

(a) What is the stock price up ratio \( u \equiv \frac{S_{t+1}}{S_t} \) corresponding to \( \epsilon_t = 1 \)? What is the down ratio \( d \)?

(b) What is the (approximate) volatility of the stock?

(c) Assume that the current price of the stock \( S_0 \) is $102 and its volatility is 30%. Assume also that the annual expected return \( \mu \) equals \( r \). Suppose that the dynamic behavior of this stock can be approximated reasonably well by the binomial process (1) if one assumes observation intervals of length 1 month, i.e., \( \delta = \frac{1}{12} \). Answer the following questions by hand calculation; rounding to cents.

i. Consider a European call option written on \( S_t \) with a strike price \( E = $120 \) and an expiration of 3 months. Determine the tree for the stock price.

ii. Using the \( S_t \) and the risk-free borrowing and lending \( \beta_t \), construct a portfolio that replicates the above option. Determine the tree for the option price.

iii. Suppose you sell 100 such calls to your customers. How would you hedge this position? Be precise.

iv. Suppose the market price of this call is $5. How would you form an arbitrage portfolio?

(d) Write a Matlab function

\[ \text{MyBinModel}(S_0, E, T, \mu, r, \sigma, M, \text{flag}) \]

to compute the premium of a European option on the stock \( S_t \) using \( O(M) \) space and \( O(M^2) \) time (Given two functions \( f(M) \) and \( g(M) \), \( f(M) = O(g(M)) \) means that there exists a constant \( \alpha > 0 \) such that \( f(M) \leq \alpha g(M) \) when \( M \to +\infty \)). The function \text{MyBinModel} returns the initial call price when flag equals one and put price when flag equals zero.

Computing the premium of the call option specified in part of (c) of the question 1 again using \text{MyBinModel} to verify its correctness.
(e) Assuming $\mu = r = 5\%$, compute the premium of the call option in (c) of question 1 for $M = 10, 100, 200$. Now letting $\mu = 30\%$ (and $r = 5\%$), compute the premium of the same call option with $M = 10, 100, 200$ respectively. How does the difference between the computed option values using different $\mu$ change with $M$? Explain your observation.

(f) Eliminating the input argument $\mu$ from MyBinModel (since the option value can be computed under the assumption $\mu = r$ assuming $M$ is sufficiently large), complete the following function

$$\text{MyBinPrice}(S_0, E, T, r, \sigma, M, \text{flag})$$

to calculate the premium of a European option.

Now assume that the stock price process is a continuous geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dX_t.$$  

A European option price can be computed exactly using the Black-Scholes formula implemented by the \texttt{blsprice} function in Matlab. Tabulate the errors of the approximate option prices computed using your \texttt{MyBinPrice} for the same call option for $M = 10 : 100 : 1000$.

(g) Now consider the same call option but with different strike prices. In one plot, graph the call option intrinsic value $\max(S_0 - E, 0)$ as a function of the strike $E$ and the call option premiums $C(E)$ computed using \texttt{MyBinPrice} for $E = 50 : 1 : 150$.

What do you observe about the relationship of the call price curve with the intrinsic function $\max(S_0 - E, 0)$? Perform the same experiment for the put option. What do you observe now?

2. (25 points) Let $X_t$ be a Wiener process. Consider the continuous price process

$$S_t = S_0 e^{(\mu - \frac{1}{2} \sigma^2)t + \sigma X_t}.$$  

(a) Using Ito’s Lemma, calculate $dS_t$.

(b) What is the expected rate of change of $S_t$?

(c) If the exponential term in the definition of $S_t$ did not contain the $\frac{1}{2}\sigma^2 t$ term, what would be the $dS_t$? What would then be the expected rate of change in $S_t$?