1. Numerical Solution

\[
\begin{align*}
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q) S \frac{\partial V}{\partial S} - r V &\leq 0, \quad V(S, t) - \text{payoff}(S) \geq 0 \\
(V(S, t) - \text{payoff}(S)) \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q) S \frac{\partial V}{\partial S} - r V \right) &= 0 \\
V(S, T) &= \text{payoff}(S).
\end{align*}
\]

Boundary conditions \(V(S, t)\) for \(S\) goes to \(+\infty\) and \(V(0, t)\) can be determined.

Let \(\delta t = \frac{T}{M}\), \(\delta S = \frac{S_{\max}}{N}\), \(t_j = (j - 1) \cdot \delta t\), and \(S_i = (i - 1) \cdot \delta S\). Let \(\{(S_i, t_j)\}_{i=1,\ldots,N, N+1}^j\) denote a uniform discretization of the region \([0, S_{\max}] \times [0, T]\). Let \(V_i^j\) denote the option value at \(t_j\) and \(S_i\). Assume that \(\{V_i^j\}\) have been computed for \(i = 1, 2, \ldots, N + 1\).

To compute the value \(V_i^{j-1}\) at time \(t_{j-1}\), let \(x = [V_1^{j-1}; \cdots; V_N^{j-1}]\) and \(g = [\text{payoff}(S_2); \cdots; \text{payoff}(S_N)]\). The finite difference formulation at time \(t_{j-1}\) can be written as

\[
Ax - b^j \geq 0, \quad x \geq g \\
(x - g) \cdot (Ax - b^j) = 0
\]

(1)

For explicit method, \(A = I\).

For Crank-Nicolson, \(A\) and \(b^j\) are given in Assignment 3. For implicit method, see Assignment 4.

Remarks.

- Assume that \(x\) solves the discretized LCP from the implicit method, it can be shown that the iterates \(\{V_i^j\}\) converge to the solution of the PDCP as \(\delta t, \delta S \to 0\);
• A general discrete LCP (which occurs in many applications)

\[ \begin{cases} 
  x \geq g, & Ax - b \geq 0 \\
  (x - g) \cdot (Ax - b) = 0
\end{cases} \]

is a difficult problem. This LCP problem has a unique solution for all \( g \) if and only if all the principal submatrices of the matrix \( A \) are positive definite.

There are many iterative methods; depending on the property of \( A \).


If the explicit finite difference method is used, the discrete LCP is

\[ \begin{cases} 
  x \geq g, & x - b^j \geq 0 \\
  (x - g) \cdot (x - b^j) = 0
\end{cases} \]

This can be solved simply as following:

\[ x = \max(b^j, g) \]

This confirms that the binomial method is right since the trinomial method corresponds to the explicit finite difference method.

The following procedure is often used to approximate American option values even when implicit/Crank-Nicolson method is used

\[ \text{Compute } V_i^{M+1}, \ i = 1, \cdots, N + 1 \]
\[ \text{for } j = M + 1 : -1 : 2 \]
\[ \text{Compute } \{ V_i^{j-1}, V_{N+1}^{j-1} \} \]
\[ \text{solve } Ax = b^j \]
\[ V_{i+1}^{j-1} = \max(x_i, \text{payoff}(S_i)), \ i = 1, \cdots, N - 1 \]
\[ \text{end} \]
Note that $x^j$ does not solve the LCP (1) since modification of $x^j$ may violates the constraint $Ax^j - b^{j-1} \geq 0$.

3. The Projected SOR Method.

An iterative method solves $Ax = b$ by generating an infinite sequence \{x^{(k)}\} according to

$$x^{(k)} = Bx^{(k-1)} + \tilde{b}$$

such that $\lim_{k \to +\infty} x^k = x^*$ and $Ax^* = b$. Consider

$$e_i x_{i-1} + d_i x_i + f_i x_{i+1} = b_i, \quad i = 1, 2, \ldots, n$$

with $e_1 = f_n = 0$. Assume that $x^{(k-1)}$ has been computed.

**Jacobi:**

$$x^{(k)}_i = \frac{b_i - e_i x_{i-1}^{(k-1)} - f_i x_{i+1}^{(k-1)}}{d_i}$$

end

**Gauss-Siedel:**

$$x^{(k)}_i = \frac{b_i - e_i x_{i-1}^{(k)} - f_i x_{i+1}^{(k-1)}}{d_i}$$

**SOR:** $0 < \omega < 2$

$$\tilde{x}_i^{(k)} = \frac{b_i - e_i x_{i-1}^{(k)} - f_i x_{i+1}^{(k-1)}}{d_i}$$

$$x_i^{(k)} = x_i^{(k-1)} + \omega (\tilde{x}_i^{(k)} - x_i^{(k-1)}) \equiv (1 - \omega) x_i^{(k-1)} + \omega x_i^{(k)}$$

Remarks:

- If the spectral radius $\rho(B) < 1$,

$$\rho(B) = \max(|\lambda| : Bx = \lambda x \text{ for some } x \neq 0)$$

then SOR converges at the speed of $O(\rho(B)^k)$; $\omega$ is chosen to minimize $\rho(B)$; It can be shown the optimal omega is in the interval $(1, 2)$.

- Each iteration costs $O(n)$. 

3
**PSOR** for $Ax - b \geq 0, \ x \geq g, \ (Ax - b) \cdot (x - g) = 0$: start with $k = 0$ and $x^{(0)} \geq g$ (note that this is important for convergence).

\[
\begin{align*}
\text{error} &= \text{inf;} \ k = 0; \\
\text{While error} &> \text{tol} \& k < \text{imax} \\
&k = k + 1 \\
&\text{for } i = 1 : n, \\
&\quad \tilde{x}^{(k)}_i = \frac{b_i - c_i x^{(k-1)}_{i-1}}{d_i} - f x^{(k-1)}_{i+1} \\
&\quad \hat{x}^{(k)}_i = x^{(k-1)}_i + \omega \left( \tilde{x}^{(k)}_i - x^{(k-1)}_i \right) \\
&\quad x^{(k)}_i = \max(\hat{x}^{(k)}_i, g_i) \\
\text{end} \\
\text{error} &= \|x^{(k)} - x^{(k-1)}\|_2^2 \\
\text{end}
\end{align*}
\]

Remarks:

- Note that the matrix $A$ is not symmetric. When $\delta t < \frac{1}{|r - q|N^2 + r}$, the matrix $A$ is strictly (row) diagonally dominant, i.e.,

\[
|a_{ii}| > \sum_{j=1, j \neq i} |a_{ij}|
\]

In addition, $a_{ii} > 0$. Thus $A$ is $H_+\text{-matrix}$. There exists $1 < \bar{\omega} \leq 2$ such that, when $\omega \in (0, \bar{\omega})$, PSOR converges to the solution of the LCP.

- Performance of PSOR depends on $\omega$ which depends on the parameters $\sigma, r, q$ etc; $\omega$ can be determined by trial and error.