CS4860-2020-Lecture-5-iPC

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Abstract

In this lecture we examine the intuitionistic propositional calculus, iPC, and contrast it with the propositional calculus presented in Smullyan's textbook for the course, *First-Order Logic*. We begin with three axiomatizations of iPC. Later we will provide an evidence semantics for these axioms.

1 Intuitionistic Propositional Logic Axioms

We start with axioms given in Joan Moschovakis' article in the *Stanford Encyclopedia of Mathematics*, SEP [2]

1. $A \Rightarrow (B \Rightarrow A)$.
2. $(A \Rightarrow B) \Rightarrow ((A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)$
3. $(A \Rightarrow (B \Rightarrow (A \ \& \ B))$
4. $(A \ \& \ B) \Rightarrow A$
5. $(A \ \& \ B) \Rightarrow B$
6. $A \Rightarrow (A \vee B)$
7. $B \Rightarrow (A \vee B)$
8. $(A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \vee B) \Rightarrow C))$
9. $(A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A)$
10. $\neg A \Rightarrow (A \Rightarrow B)$

Next we give Heyting's eleven axioms for iPC [1].

1. $P \Rightarrow (P \vee Q)$.
2. $(P \vee Q) \Rightarrow (Q \vee P)$
3. $((P \Rightarrow R) \;\&\; (Q \Rightarrow R)) \Rightarrow ((P \vee Q) \Rightarrow R)$
4. $(P \Rightarrow (P \;\&\; P)$
5. $((P \Rightarrow Q) \;\&\; (Q \Rightarrow R)) \Rightarrow ((P \vee Q) \Rightarrow R)$
6. $Q \Rightarrow (P \Rightarrow Q)$
7. $(P \;\&\; (P \Rightarrow Q)) \Rightarrow Q$
8. $(P \;\&\; Q) \Rightarrow (Q \;\&\; P)$
9. $(P \Rightarrow Q) \Rightarrow ((P \;\&\; R) \Rightarrow (Q \;\&\; R)$
10. $\neg P \Rightarrow (P \Rightarrow Q)$
11. $((P \Rightarrow Q) \;\&\; (P \Rightarrow \neg Q)) \Rightarrow \neg P$
12. $((P \Rightarrow R) \;\&\; (Q \Rightarrow R)) \Rightarrow ((P \vee Q) \Rightarrow R)$

Finally we present the axioms given Stephen Kuznetsov, a student at the University of Pennsylvania interested in intuitionistic logic.

1. $A \Rightarrow (B \Rightarrow A)$.
2. $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$
3. $(A \;\&\; B) \Rightarrow A$
4. $(A \;\&\; B) \Rightarrow B$
5. $(A \Rightarrow (B \Rightarrow (A \;\&\; B))$
6. $(A \Rightarrow (A \vee B)$
7. $(B \Rightarrow (A \vee B)$
8. $(A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \vee B) \Rightarrow C)$
9. $\bot \Rightarrow A$
10. There is one *inference rule*: $A \;\; (A \Rightarrow B)/B$

We now present the functional programs that are said to *realize* these axioms.

1. $A \Rightarrow (B \Rightarrow A)$ by $\lambda(a.\lambda(b.a))$
2. $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$
   by $\lambda(f.\lambda(ab.\lambda a.f(a)(ab(a))))$
3. $(A \,\&\, B) \Rightarrow A$ by $\lambda(ab.first(ab))$
4. $(A \,\&\, B) \Rightarrow B$ by $\lambda(ab.second(ab))$
5. $(A \Rightarrow (B \Rightarrow (A \,\&\, B))$ by $\lambda(a.\lambda(b. < a, b >))$
6. $(A \Rightarrow (A \vee B)$ by $\lambda(a.inl(a))$
7. $(B \Rightarrow (A \vee B)$ by $\lambda(b.inr(b))$
8. $(A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \vee B) \Rightarrow C)$ by $\lambda(f.\lambda(g.\lambda(ab.decide(ab.a.f(a); b.g(b)))))$
9. $\perp \Rightarrow A$ by $\lambda(x.any(x))$
10. There is one *inference rule*: $A \;\; (A \Rightarrow B)/B$

References

[1] A. Heyting. *Intuitionism, An Introduction.* North-Holland, Amsterdam, 1966.

[2] Joan Rand Moschovakis. Intuitionistic logic. *Stanford Encyclopedia of Philosophy*, 2018.