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Historical introduction and fundamental notions[1]

The gradual transformation of the mechanism of mathematical thought is a consequence of the modifications which, in the course of history, have come about in the prevailing philosophical ideas, firstly concerning the origin of mathematical certainty, secondly concerning the delimitation of the object of mathematical science. In this respect we can remark that in spite of the continual trend from object to subject of the place ascribed by philosophers to time and space in the subject–object medium, the belief in the existence of immutable properties of time and space, properties independent of experience and of language, remained well-nigh intact far into the nineteenth century. To obtain exact knowledge of these properties, called mathematics, the following means were usually tried: some very familiar regularities of outer or inner experience of time and space were postulated to be invariable, either exactly, or at any rate with any attainable degree of approximation. They were called axioms and put into language. Thereupon systems of more complicated properties were developed from the linguistic substratum of the axioms by means of reasoning guided by experience, but linguistically following and using the principles of classical logic. We will call the standpoint governing this mode of thinking and working the *observational* standpoint, and the long period characterized by this standpoint the observational period. It considered logic as autonomous, and mathematics as (if not existentially, yet functionally) dependent on logic.

For space the observational standpoint became untenable when, in the course of the nineteenth and the beginning of the twentieth centuries, at the hand of a series of discoveries with which the names of Lobatchefsky, Bolyai, Riemann, Cayley, Klein, Hilbert, Einstein, Levi-Cività and Hahn are associated, mathematics was gradually transformed into a mere science of numbers; and when besides observational space a great number of other spaces, sometimes exclusively originating from logical speculations, with properties distinct from the traditional, but no less beautiful, had found their arithmetical realization. Consequently the science of classical (Euclidean, three-dimensional) space had to continue its existence as a chapter without priority, on the one hand of the aforesaid (exact) science of numbers, on the other hand (as applied mathematics) of (naturally approximative) descriptive natural science.

In this process of extending the domain of geometry, an important part had been played by the *logico-linguistic method*, which operated on words by means of logical rules, sometimes without any guidance from experience and sometimes even starting from axioms framed independently of experience. Encouraged by this the *Old Formalist School* (Dedekind, Cantor, Peano, Hilbert, Russell, Zermelo, Couturat), for the purpose of a rigorous treatment of mathematics *and logic* (though not for the purpose of furnishing objects of investigation to these sciences), finally rejected any elements extraneous to language, thus divesting logic and mathematics of their essential difference in character, as well as of their autonomy. However, the hope originally fostered by this school that mathematical science erected according to these principles would be crowned one day with a proof of its non-contradictority was never fulfilled, and nowadays, after the logical investigations performed in the last few decades, we may assume that this hope has been relinquished universally.

Of a totally different orientation was the *Pre-intuitionist School*, mainly led by Poincaré, Borel and Lebesgue. These thinkers seem to have maintained a modified observational standpoint for the introduction of natural numbers, for the