The No-Retraction Theorem

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1 Rational Cube Complexes

All vectors v, p, q, etc. will be in $\mathbb{R}^k$ for some fixed k. All the coordinates of a *rational vector* in $\mathbb{Q}^k$ are rational numbers. For rational numbers r and s we can decide $r = s$, $r < s$, and $r \leq s$. A *rational cube* $c = \mathrm{Cube}(a, b)$ is given by two rational vectors $a, b \in \mathbb{Q}^k$. Vector p is in the cube c if $a_i \leq p_i \leq b_i$ for all $i < k$. So, cube c is *inhabited* (i.e. non-empty) just if $a_i \leq b_i$ for all $i < k$.

The i^{th} interval $[a_i, b_i]$ has dimension 1 if $a_i < b_i$ and has dimension 0 if $b_i \leq a_i$. The dimension of cube c is the sum of the dimensions of each of its intervals, so $0 \leq \dim(c) \leq k$. The *faces* of interval $[a, b]$ are the two 0-dimensional intervals $[a, a]$ and $[b, b]$ and interval $[a, b]$ itself. A cube f is a *face* of cube c if, for all $i < k$, the i^{th} interval of f is a face of the i^{th} interval of c. We write $f \leq c$ when f is a face of c. An n-dimensional cube c has exactly $2n$ faces with dimension $n - 1$, obtained by replacing just one of the 1-dimensional intervals of c by one of its 0-dimensional faces.

An n *-dimensional rational cube complex* (n-dim complex, for short) is a finite list K of rational cubes, with no repeats, such that each cube $c \in K$ is inhabited and has dimension n, and for any two cubes c and d in K, if they have a point in common (i.e. they overlap) then there is a common face $f \leq c$, $f \leq d$ such that the intersection of c and d is exactly f. So, we say that the cubes in K overlap only along faces.

The *boundary*, ∂K, of the n-dim complex K is the list of $n-1$ dimensional cubes f that are faces of an *odd number* of cubes in K. Then ∂K is an $n-1$-dimensional complex.

A cube d is a *half-cube* of cube d if for each interval $[a_i, b_i]$ of c, the corresponding interval of d is either $[a_i, \frac{a_i+b_i}{2}]$ or $[\frac{a_i+b_i}{2}, b_i]$. Each n-dimensional cube c has 2^n half cubes. The *sub-division* K' of complex K is the list of all the half cubes of K.

2 Statement of the theorem

The *polyhedron*, $|K|$ of the complex K is the set

$$\{p : \mathbb{R}^k \mid \neg\neg(\exists c \in K.\ p \in c)\}$$

Recall that for vectors p, q in $\mathbb{R}^k$ we write $p \equiv q$ if the distance $d(p,q) = 0$ where $d(p,q)$ can be any of the metrics $\Sigma|p_i - q_i|$, $max\{|p_i - q_i|, i < k\}$, or $\sqrt{\Sigma(p_i - q_i)^2}$.

A *boundary retraction* for K is a *function* r in $|K| \to |\partial K|$ such that for all $x \in |\partial K|$, $r(x) \equiv x$. Recall that r is a *function* if $x \equiv y \Rightarrow r(x) \equiv r(y)$. We write $\text{Retract}(|K| \to |\partial K|)$ for the type of boundary retractions for K.

Theorem 1 *(No-retraction) For every $n \geq 0$, for every n-dimensional complex K,*

$$\textit{length}(K) > 0 \Rightarrow \neg\textit{Retract}(|K| \to |\partial K|)$$

Here are some facts we will need for the proof of this theorem:

Lemma 1 *If A and B are compact subsets of $\mathbb{R}^k$ then the* stable union *$A \mathbin{\dot\cup} B$ is compact, where $A \mathbin{\dot\cup} B = \{p : \mathbb{R}^k \mid \neg\neg(p \in A \vee p \in B)\}$.*

Lemma 2 *$|K|$ is compact.*

Lemma 3 *$\partial(\partial K) = \emptyset$.*

Lemma 4 *$|K'| = |K|$.*

Lemma 5 *$\partial(K') = (\partial K)'$.*

We also need Brouwer's theorem:

Theorem 2 *Every* function *from a compact metric space X to a metric space Y is* uniformly continuous*.*

3 Easy cases of the theorem

We prove the theorem by induction on the dimension n. When $n = 0$ the theorem is easy because $|K| \neq \emptyset$ but $|\partial K| = \emptyset$. So there can't be any map from $|K|$ to $|\partial K|$.

We need a special argument for the case $n = 1$. In this case each "cube" in K has just one coordinate i where the interval $[a_i, b_i]$ has $a_i < b_i$. We proceed by induction the length of K, i.e. on the number of cubes in K. This number has to be at least one. If it is exactly one, then the retraction has to map the interval $[a_i, b_i]$ to its endpoints, keeping the endpoints fixed. Let $\epsilon = (b_i - a_i) > 0$ and let $m = \frac{a_i + b_i}{2}$ be the midpoint. Then the function $f(t) =$

$r(t) - m$ has $f(a_i) < 0$ and $f(b_i) > 0$ so by the approximate intermediate value theorem there is a t where $|f(t)| < \epsilon/2$. But then $f(t)$ can't be either a_i or b_i.

If there is more than one "cube" in K, then we choose one interval cube, call it I, that contains an endpoint p in $|\partial K|$. If both endpoints p and q of I are in $|\partial K|$ then we have the same contradiction as in the case when K is a single interval cube. Otherwise the boundary of $K - I$ will be $\partial K - \{p\} + \{q\}$.

We can define a map h that sends point p to point q and leaves all the other points of $|\partial K|$ fixed. Then if we had a retraction r from $|K|$ to $|\partial K|$ we could compose $h \circ r$ (map r followed by map h) to get a retraction from $|K - I|$ to $\partial(K - I)|$. This contradicts our induction hypothesis since the length of $K - I$ is less than the length of K.

4 The main case of the theorem

We now have $n > 1$ and the induction hypothesis that there are no boundary retractions for non-empty complexes of dimension $n - 1$. We suppose that we have a retraction $r : |K| \to |\partial K|$ and prove False.

Here are the steps of the proof:

1. Choose $c \in \partial K$.
2. Choose $p \in c$ at the 1/3 point of each dimension.
3. Choose M so that $d(p, |K - \{c\}|) > 1/M$.
4. Find $\delta > 0$ so that $d(x, y) < \delta \Rightarrow d(r(x), r(y)) < 1/2M$
5. Subdivide K enough times that every cube in K' is smaller than δ.
6. Find $t \in K'$ with $p \in t$. (Then p is at 1/3 or 2/3 of t).
7. Find $J \geq 2M$ so that $d(p, |K' - t|) > 1/J$
8. Let $RN(e) \Leftrightarrow \exists x \in e.\ d(r(x), p) < 1/J$.
9. Let $L = \{e \in K' \mid RN(e)\}$.
10. Note $r(|L|) \subseteq |c|$.
11. Note $RN(f) \Rightarrow (f \in \partial L \Leftrightarrow f \in \partial K)$.
12. Let $S = \partial L - \{t\}$. Note $\dim(S) = n - 1$.
13. Note $d(p, r(|S|)) \geq 1/J$.
14. $\partial S = (\partial(\partial L) \text{ xor } \partial t) = \partial t = \text{faces of } t$
15. $|S| \subset |L|$, so $r(|S|) \subset |c| - \{x \mid d(x, p) < 1/J\}$

16. There is a retraction h from $|c| - \{x \mid d(x,p) < 1/J\}$ to |faces of t|.

17. Then $h \circ r$ is a boundary retraction for S, which contradicts the induction hypothesis.

QED. !! (Lots of stuff to check in the above proof, but all of it completely checked using Nuprl.)

5 Brouwer's fixedpoint theorem

From the no-retraction theorem for complexes K we get the corollary that for a single rational cube c there is no retraction from $\{x : \mathbb{R}^k \mid x \in c\}$ to |faces of c| . From this we can get that there is no-retraction from the unit ball $B(k) = \{x : \mathbb{R}^k \mid |||x||| \leq 1\}$ to its boundary $\{x : \mathbb{R}^k \mid |||x||| = 1\}$ by constructing a *homeomorphism* that takes the unit rational cube to the unit ball.

As we discussed last time, we derive from that no-retraction theorem Brouwer's fixedpoint theorem:

Theorem 3 *For every* function $f \in B(k) \to B(k)$*, and every* $\epsilon > 0$*, there exists* $x \in B(k)$ *such that* $d(f(x), x) < \epsilon$*.*