CS4860-2019fa
Smullyan-FOL-Completeness-Summary

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September 25, 2019

Abstract

Our textbook for the course, *First-Order Logic* by Raymond Smullyan provides a very clear and detailed account of the *completeness theorem* for First-Order logic, a result first proved by Gödel in his 1929 PhD thesis, published in 1930. There is an English translation in the book *From Frege to Gödel* [3]. This comprehensive book also includes an English translation of Gödel's remarkable *incompleteness* theorem for *Principia Mathematica* [4] for which Gödel is most famous. We will examine incompleteness in due course. Smullyan's proof of FOL completeness is exceptionally clear, and he became quite well known for it and his textbook.

Smullyan's completeness proof uses König's Lemma from page 32 of Smullyan. We have included Crystal Cheung's MEng thesis as a course resource. She discusses this lemma and relates it to Brouwer's Fan Theorem which we will examine later in the course.

In the next segment of the course, we will examine an intuitionistic version of First-Order Logic, which we abbreviate as *iFOL*. This logic can also be seen as a programming language as we will learn. It turns out that completeness for iFOL was a hard problem, open for several years. Mark Bickford and I solved this problem in 2014 [1], and we will briefly discuss this proof later in the course.

Smullyan also discusses the famous Löwenheim theorem on page 61. It says that if a formula X is satisfiable, then it satisfiable in a denumerable domain. In this section he also discusses more practical proof procedures. This is an interesting topic for applied logic because we want to understand how to create the most readable and clear FOL proofs.

On page 61 near the bottom, Smullyan discusses a more efficient proof procedure that orders the application of the four kinds of rules. He suggests first using all of the α and δ points and then the β points, and finally the γ points. It is a simple exercise to see why this ordering is good. Thinking about this exercise helps understand the proof method.

1 Historical Context

There is a thread in the foundations of mathematics originating with Plato (415 BC to 369 BC) and his Academy in Athens (387 BC until 529 AD when it was closed by the Emperor Justinian). Plato believed in truths that we can see with the *eye of the mind.* That belief underlies the philosophy of *Platonism.* Aristotle, a student of Plato's Academy in Athens, regarded logic as preliminary to science and philosophy and applicable to all reasoning. He said this about two thousand four hundred years ago. More recently, about one hundred years ago, L.E.J. Brouwer also expressed in detail his belief that mathematics is grounded in our *intuitions about numbers, algorithms, space, and most critically,* time. Brouwer believed that mathematical objects are constructed mentally, starting with a few fundamental concepts such as natural numbers, points, and lines. The relationship of the progression of time and of the continuity of lines to the real numbers was of special interest to Brouwer. We will examine this relationship in detail because his insights and results convinced us to fully integrate his ideas into the logic of the Nuprl proof assistant.[1]

One of the mostly widely known Greek mathematicians is Euclid. He lived in Alexandria circa 300 BCE where he wrote his *Elements.* This classic remains widely read and studied, to learn both geometry and logic. What is hard to believe is that we continue to deepen our knowledge of Euclidean geometry, now using proof assistants [2]. [2]

References

[1] Robert Constable and Mark Bickford. Intuitionistic Completeness of First-Order Logic. *Annals of Pure and Applied Logic*, 165(1):164–198, January 2014.

[2] Ariel Kellison, Mark Bickford, and Robert Constable. Implementing Euclid's straightedge and compass constructions in type theory. *Annals of Mathematics and Artificial Intelligence*, Sep 2018.

[3] J. van Heijenoort, editor. *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931.* Harvard University Press, Cambridge, MA, 1967.

[4] A.N. Whitehead and B. Russell. *Principia Mathematica*, volume One, Two, Three. Cambridge University Press, 2nd edition, 1925–27.

[1] As of 2019, as far as we know, the Nuprl proof assistant is the only one that has fully implemented Brouwer's remarkable insights.

[2] Euclid's proposition 117 of Book X is essentially that $\sqrt{2}$ is irrational, but it was not included in Euclid's original text, so it is nowadays omitted. Until this discovery, the Pythagoreans identified numbers with geometry.