Constructive Intermediate Value and Fixed Point Theorems

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Abstract

Errett Bishop did not state or prove a version of the *intermediate value theorem* (IVT) in his book *Foundations of Constructive Analysis.* In the revised and extended book that he wrote with Douglas Bridges, *Constructive Analysis*, they added a weak constructive version of this theorem, one that produces an approximate value within a small epsilon, ϵ, of a "true" intermediate value. We prove this result following their book and discuss how to use similar ideas to prove an *epsilon version* of *Brouwer's fixed point theorem* on intervals of the reals using an approximate constructive IVT. We could call this a *weak fixed point* or an *approximate fixed point.* We provide a simple picture to illustrate why the classical intermediate value theorem is not constructively justified.

This article also addresses the value of creating better concepts and paradigms for discussing what we now call "constructive mathematics" versus "classical mathematics". These are not separate subjects, and our work on this topic stresses the integration of these aspects of mathematics. It also reveals how close they come to *infinite precision* "numerical analysis." One goal of mathematical logic is to facilitate the integration of mathematical theories and discover unexpected fundamental connections among them.

The work of Bishop and Bridges and our implementation of many of their ideas is illustrative of the broader theme of exploring the *computational aspects of mathematics* in virtually all of its subareas. This has been a trend for centuries, now accelerated by broad applications of computer science to mathematics and mathematics to computer science.

1 Introduction

There is considerable interest in implementations of Brouwer's fixed point theorem. This can be seen by asking Google about constructive fixed point theorems. The connection is that a constructive proof would provide an implementation, a way of computing fixed points. It is interesting that this question about the fixed point theorem is related to whether there is a constructive proof of an intermediate value theorem. That would presumably provide a way of computing the intermediate value.

The American mathematician Errett Bishop wrote an entire book on how to do real analysis constructively, entitled *Foundations of Constructive Analysis* [5]. But he did not provide a constructive version of the *intermediate value theorem*, a topic related to the fixed point theorems. Later he was joined by Douglas Bridges in revising and extending the book, entitled *Constructive Analysis* [6]. Many new topics were added, especially on integration, and also a version of the intermediate value theorem and its computational content. The constructive version does not provide a way to compute the intermediate value to arbitrary precision, it only provides an ϵ approximation. From this it is possible to obtain ϵ approximations of Brouwer's fixed point theorem as Veldman has shown in his article *Perhaps the Intermediate Value Theorem* [27].

Beyond the practical value of having constructive fixed point theorems, there is a purely intellectual and philosophical interest in this topic because Brouwer is known not only for his fixed point theorems and for advancing topology and homotopy theory, he is also known for proposing a broad reform of mathematics. For example he proposed modifications of the standard laws of logic that go back to Aristotle, famously denying the "law of excluded middle" that asserts that every mathematical proposition is either true or false. He even objected to the hard won definition of the real numbers, and proposed an alternative that allows computing them to any degree of precision. In the course of this work he was in a position to deny the truth of basic results such as the intermediate value theorem of calculus.

The constructive intermediate value theorem which is given in half a page in *Constructive Analysis* was formalized by Mark Bickford in constructive type theory using the Nuprl proof assistant. Printed out this proof would take dozens of pages. Here the proof is summarized briefly as an introduction to the on-line fully formalized proof. This proof can be used to give an approximate fixed point theorem that is fully formalized.

Errett Bishop was keenly aware of Brouwer's writings on intuitionistic mathematics and their relevance to real analysis. But he did not accept Brouwer's notion of *free choice sequences*, nor his *continuity principle*, nor his *bar induction principle*. He penned this dismissal of these ideas saying: "*In Brouwer's case there seems to have been a nagging suspicion that unless he personally intervened to prevent it, the continuum would turn out to be discrete.*" Bishop also admired Brouwer and said: "Brouwer fought the advance of formalism and undertook the disengagement of mathematics from logic. He wanted to strengthen mathematics by associating with every theorem and every proof a pragmatically meaningful interpretation. His program failed to gain support." Even though Brouwer was not alive to read these words, they ring hollow because there was at the time and remains today a very talented and lively group of intuitionistic mathematicians who continue to advance Brouwer's approach to mathematics and broaden his strong impact on logic.

Brouwer was not a logician, yet he had a profound impact on logic. The same can be said of Leibniz, and in one of our articles [8] about the Nuprl *proof assistant*, a machine that helps us create formally correct proofs, we noted the deep understanding Leibniz had about the role of logic in mathematics. We quote those words again now in a footnote, as before (we added the parenthetical words).[1]

The discipline of computer science (CS) has completely changed the context in which intuitionism and constructive mathematics are developing. First, CS has made formal logics acceptable and useful because computers implement them to do important work, massive amounts of work on which modern society depends. Hundreds of software systems are now vital, many are critical. All are built using programming languages and "artificially intelligent machines." Computer science has also given us *a new common language* for talking about mathematics. Instead of discussing constructive mathematics versus intuitionistic mathematica versus classical mathematics, the issue has become, what can we implement efficiently in software. In this time, the language of computing is nearly as universal as the language of mathematics, and it makes good sense to ask, what results in mathematics can be made computable and efficiently computable.

One advantage of this new common core at the heart of science is that there is a way to naturally integrate deep ideas from philosophy, logic, mathematics, computer science, information science, cognitive science and beyond. They become part of a broad theory of that lies at the heart of computer science yet is relevant to all of science and engineering.

1.1 Background on constructive and intuitionistic analysis

Errett Bishop was keenly aware of L.E.J. Brouwer's writings on intuitionistic mathematics and their relevance to real analysis. But he did not accept Brouwer's notion of *free choice sequences*, nor his *Continuity Principle*, nor his *Bar Induction* principle. He penned this unforgettable dismissal of these ideas saying: "*In Brouwer's case there seems to have been a nagging suspicion that unless he personally intervened to prevent it, the continuum would turn out to be discrete.*" Bishop also admired Brouwer and said: "Brouwer fought the advance of formalism and undertook the disengagement of mathematics from logic. He wanted to

[1]We can judge immediately (even mechanically) whether propositions presented to us are proved, and that which others could hardly do with the greatest mental labor and good fortune, we can produce with the guidance of symbols alone ... as a result of this, we shall be able to show within a century what many thousands of years would hardly have granted to mortals otherwise [17].

strengthen mathematics by associating with every theorem and every proof a pragmatically meaningful interpretation. His program failed to gain support." Even though Brouwer was not alive to read these words, they ring hollow because there was at the time and still is a very talented and lively group of intuitionistic mathematicians who continue to advance Brouwer's approach to mathematics and his strong impact on logic. The logic of the Nuprl proof assistant is now a fully intuitionistic logic [?], perhaps at present the only implemented fully intuitionistic theory, but in due course there will others.

The discipline of computer science (CS) has completely changed the context in which intuitionism is developing. First, CS has made formalism acceptable and useful because computers implement formalisms to do important work, massive amounts of work on which modern society depends. Hundreds of software systems are now vital, many are critical. All are built using programming languages and "artificially intelligent machines."

Formal accounts of Brouwer's ideas have made them practically useful. A key player in this change was the American logician Stephen C. Kleene and his students. His book with Vesley, *Foundations of Intuitionistic Mathematics* [15], made Brouwer's ideas precise and accessible. The writing was so exact and detailed that computer scientists at Cornell could read it and implement the concepts, which we did [3]. Other important writings by logicians and intuitionistic mathematicians significantly contributed to our understanding [1, 2, 14, 21, 22, 20, 26, 25, 28, 29, 24]. Computer scientists at Cornell and in other research groups have formalized many definitions and theorems from Bishop [5] and from Bishop and Bridges [6] and are integrating them with ideas from Brouwer and many of the intuitionist mathematicians cited just above. Also, computer scientists needed to understand concepts almost identical to Brouwer's free choice sequences because they explain the behavior of distributed systems and ground the logics used to reason about them such as *event logics* [30, 4, 16, 31, 3, 19].

1.2 The classical intermediate value theorem

First we give the short classical proof from Courant.

Classical Intermediate Value Theorem: A function f(x) continuous on a closed interval $a \leq x \leq b$, which is negative for $x = a$ and positive for $x = b$, or conversely, assumes the value 0 at least once in the interval.

Proof: In the interval there are an infinite number of points for which $f(x) < 0$, on account of the continuity of the function, in fact this is true for a whole interval beginning at the point a. The set of those points x for which $f(x) < 0$ has a least upper bound ξ, which is greater than a. Since in every neighborhood of ξ, for which $f(x) < o$, we must have $f(\xi) \leq 0$ (whence in particular $\xi \neq b$). It is impossible, however, that $f(\xi) < 0$, for then $f(x)$ would be negative in a sufficiently small neighborhood of ξ, including values x greater than ξ, in contradiction to the hypothesis that ξ is the upper bound of the values x for which $f(x) < 0$. Therefore $f(\xi) = 0$, and our assertion is proved. **Qed**

Courant says that the following is a slight generalization of the theorem.

If we assume that f(a) = α and f(b) = β, and if ν is any value between α and β, the continuous function f(x) assumes the value ν at least once in the interval.

For the continuous function $\phi(x) = (f(x) - \nu)$ will have different signs at the two ends of the interval, and will therefore assume the value 0 somewhere in the interval.

2 The constructive and intuitionistic real numbers

For Bishop a *constructive real number* is a computable sequence of rational numbers, say (x_n) such that $|x_m - x_n| \leq 1/m + 1/n$. A *computable sequence* means the following verbatim p.15 "The dependence of one quantity on another is expressed by the basic notion of an operation. An operation from a set A into a set B is a finite routine f which assigns an element $f(a)$ of B to each given element a of A." This idea of a "finite routine" has been made precise by logicians studying computability, going back to Church and Turing (neither cited by Bishop) [7, 23].

We denote the type of constructive real numbers by $\mathbb{R}$ and the type of compact intervals (defined below) by $\mathbf{\Pi}$.

The intuitionistic real numbers are sequences of rationals that converge as defined just above, but they can be *freely generated*, i.e. *chosen* by an agent without following a rule or an algorithm. The values are "freely chosen" by an agent to produce an unbounded sequence. Brouwer called this agent the *creative subject*[24].

3 Continuity and the constructive weak Intermediate Value Theorem (wIVT)

The notion of continuity used by Bishop and later by Bishop and Bridges is not pointwise continuity. Bishop discusses this issue in his first book at the end of Chapter 1. He says:"The concept introduced in definition 4.5 is classically called *uniform* continuity. We abbreviate this to *continuity*, since the classical version of continuity (that is pointwise continuity) is not used anywhere in this book. Classically, one proves that on a compact interval, pointwise continuity implies uniform continuity. Constructively, there is no known proof of this result, and so we assume uniform continuity from the start."

To state a wIVT we need to define the notion of a *compact interval*; it is a nonvoid, closed, finite interval. We need to define these concepts. *Nonvoid* simply means we can construct an element of the interval. Here are the other definitions. Note, we will also refer to this result as IVTϵ.

A *closed interval* is denoted $[a, b]$, and it consists of the reals x such that $a \leq x \leq b$.
An *open interval* is denoted (a, b), and it consists of the reals x such that $a < x < b$.
A half open *on the left* interval is denoted $(a, b]$. A half open *on the right* interval is denoted $[a, b)$.

If $a \leq b$ then the intervals are said to be *proper*. We add here that intervals $[a, a]$ are degenerate.

There are also *infinite intervals* with the obvious definitions. They are these:

$(-\infty, b)$ $(-\infty, b]$ (a, ∞) $(a, \infty]$

Definition A *finite interval* is called *compact* if and only if it is nonempty and closed.

Recall that we use $\mathbf{\Pi}$ to denote the type of compact intervals.

Definition Given real numbers a, b with $a < b$, a function $f : [a, b] \to \mathbb{R}$ is called (uniformly) *continuous* if and only if there is an *operation* ω from $[a, b]$ into $\mathbb{R}$, called the *modulus of continuity* such that for each ϵ greater than 0, $\omega(\epsilon) > 0$ and $|f(x) - f(y)| < \omega(\epsilon)$ whenever $|x - y| < \epsilon$.

In classical mathematics this notion is called *uniform continuity*. The notion of *pointwise continuity* from classical mathematics is not useful because there is no known proof that a function which is continuous on a compact interval is uniformly continuous, a major result of classical analysis.

We can now prove an ϵ-version of the Intermediate Value Theorem. We call it the Weak Intermediate Value Theorem because we cannot find a point y such that $f(x) = y$. We only get "within ϵ" of y.

In type theory we can define the *set type* over a type A as $\{x : A | P(x)\}$ which is the type of all elements of A for which we can prove $P(x)$. The type does not retrain the proof of $P(a)$ but we know it has been found. There are two of these set types we can use next to define the *greatest* *lower bound*, glb, and *least upper bound*, lub, of a type with an order relation such as $\Re$.

Definition 4.2 A non empty type A of real numbers is *bounded above* iff we can find a real number a such that for all x in A, we know $x \leq a$. Such a bound is a *least upper bound*, lub, of A iff for each positive ϵ we can find an element x of A such that $x > a - \epsilon$.

A non empty type A of real numbers is *bounded below* iff we can find a real number b such that for all x in A, we know $x \geq b$. Such a bound is a *greatest lower bound*, glb, of A iff for each positive ϵ we can find an element x of A such that $x < b - \epsilon$.

We need the following lemma to prove even the Weak Intermediate Value Theorem.

Proposition 4.6 If $f : [a, b] \rightarrow \mathbb{R}$, then the values $sup\{f(x)|x : [a, b]\}$ and $inf\{f(x)|x : [a, b]\}$ exist.

Bishop and Bridges say (on page 40) that classically a continuous function maps an interval to an interval. They do not discuss the constant function f(x) = c on [0,1]. Normally we do not take [c,c] to be an interval. Any constant function on an interval is clearly continuous, and a special case.

The version in Bishop and Bridges is this.

Let f be a continuous map defined on an interval I, and let a and b be points of I, with $f(a) < f(b)$. Then for each y in $[a, b]$ and each $\epsilon > 0$, there exists x in $[min\{a, b\}, max\{a, b\}]$ such that $|f(x) - y| < \epsilon$.

Mark Bickford has given a completely formal proof in Nuprl.The proof is posted on the PRL web page, and is accessible by clicking on the image of the Bishop and Bridges book. We include that statement here after this more informal proof. Recall that $\mathbf{II}$ is the type of compact intervals.

Theorem 4.8 Weak Intermediate Value Theorem:

$$(\forall I : \mathbf{II}.\forall f : (I \rightarrow \mathbb{R}).(\forall a, b : I(a \neq b)\&(f(a) < f(b))) \Rightarrow \forall y : [f(a), f(b)]) \Rightarrow$$

$$\forall \epsilon : \mathbb{R}.((\epsilon > 0) \Rightarrow (\exists x : [min\{a, b\}, max\{a, b\}].|f(x) - f(y)| < \epsilon).$$

A proof of the theorem builds a realizer called *close* with the following type.

close: $l, u : \mathbb{R} \to (f : [l, u] \to \mathbb{R}).(\exists a, b : [l, u].f(a) < f(b)) \Rightarrow (\forall y : [f(a), f(b)].\forall \epsilon : \mathbb{R}.((\epsilon > 0) \Rightarrow \exists x : [min\{a, b\}, max\{a, b\}].|f(x) - y| < \epsilon)$.

The realizer for the weakIVT is *close* which appears in the conclusion as:

$$|f(close(l, u, f, \epsilon, y)) - y| < \epsilon.$$

The assumption that we can find a, b with $f(a) < f(b)$ rules out the case where there is a value c in such that $f(x) = c$ for all x in the interval I. Constant functions on an interval are clearly continuous.

Since $f(a) < f(b)$, we know that $\neg(a = b)$. We actually know $a \neq b$ in the strong form that either $a < b$ or $a > b$. Let m be $inf\{f(x) - y|x : [a, b]\}$ We know m exists from Prop 4.6. Suppose $m > 0$, we will show that this is impossible. That means $m = 0$.

Here is the exact formal statement proved.

$$\forall I : Interval.\forall f : I.(f[x]\ continuous\ for\ x\epsilon I) \Rightarrow$$
$$(\forall a, b : \{x : \mathbb{R}|x\epsilon I\}.((f(a) < f(b)) \Rightarrow$$
$$(\forall y : \{x : \mathbb{R}|x\epsilon[f(a), f(b)]\}.\forall e : \{z : \mathbb{R}|r0 < z\}.$$
$$\exists z : \{x : \mathbb{R}|x\epsilon[rmin(a; b), rmax(a; b)]\}.(|f(z) - y| < e))))).$$

The diagram below shows why we cannot find the exact intermediate value.

We can imagine the dotted line .... being so close to the x axis that we cannot decide whether it lies above or below. If we could find the point where the line crosses the x axis then we could decide whether the line .... was above or below. This is not decidable as it would be if the IVT were constructive.

The classical intermediate value theorem is proved in almost every textbook on analysis. Some proofs are very short as in R. Courant's classic two volume *Differential and Integral Calculus* [9]. We gave that proof above.

The constructive proof of a weak Intermediate Value Theorem in Bishop and Bridges *Constructive Analysis* [6] takes only half a page, but it is very dense as can be seen from the formalization by Mark Bickford in Nuprl. That proof is accessible from the image of the Bishop and Bridges book displayed in the Nuprl Mathematics Library. The proof in the book takes

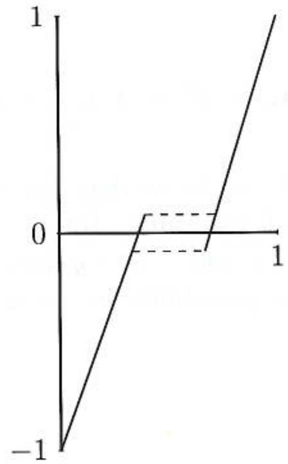

Figure 1: Example

less than half a page, but it is extremely dense as can be seen from the fact that the full Nuprl proof if written out would take many pages. The complete formal proof is available on-line at the PRL project web page, www.nuprl.org at this address:

{http://nuprl.org/MathLibrary/ConstructiveAnalysis/Constructive_Analysis.}

Here is the informal constructive proof from Bishop and Bridges.

(4.8) **A Constructive Intermediate Value Theorem**.

(4.8) Theorem: Let f be a (uniformly) continuous map on an interval I and let a and b be points in I with $f(a) < f(b)$. Then for each y in $[f(a), f(b)]$ and each $\epsilon > 0$, there exists x in $[min\{a,b\}, max\{a,b\}]$ such that $|f(x) - y| < \epsilon$.

Proof:

Since f is continuous and we assume that $f(a) < f(b)$, we must have that points a and b are separated, $a \neq b$. This means that either $a < b$ or $b < a$. Assume that $a < b$. For any y in $[f(a), f(b)]$ and any $\epsilon > 0$, consider the values $|f(x) - y|$ for all the x in the interval $[a, b]$.

Let m be the value $inf\{|f(x) - y| : a \leq x \leq b\}$. We know this exists from Theorem 4.6 above. Suppose $m > 0$. Then $(f(a) - y) \leq -m$ and $(f(b) - y) \geq m$. Let ω be a modulus of continuity for f on $[a, b]$, and choose points x_i as follows.

$a = x_0 \leq x_1 \leq ... \leq x_n = b$ such that $x_{k+1} - x_k \leq \omega(m)$ for k in the interval $0 \leq k \leq n$. Then for such a k we know this equality:

$$|f(x_{k+1} - y - (f(x_k) - y)| = |f(x_{k+1} - f(x_k)| \leq m.$$

Since $|f(x) - y| \geq m$ for all m in $[a, b]$, we know that $f(x_k) - y$ and $f(x_{k+1} - y$ are either both positive or both negative. Therefore all the values $f(x_i) - Y$ for $0|leqi \leq n$ are either all positive or all negative. Hence, $f(a) - y$ and $f(b) - y$ are either both positive or both negative. This contradiction guarantees that the possibility $m > 0$ is excluded, so that $m < \epsilon$ as required for the conclusion.

Qed

Bishop and Bridges point out that for many of the common functions f in analysis, the conclusion can be strengthened to provide an exact value, $f(x) = y$ for some x in $[min\{a, b\}, max\{a, b\}]$. For example this happens if the function f is strictly increasing.

4 Constructive Versions of Brouwer's Fixed Point Theorems

Brouwer's fixed point theorems are well known, useful, and in some way "iconic," but his proofs were developed and published at a time when he temporarily "suspended" adherence to his own intuitionistic mathematics. Perhaps this was done to establish himself as one of the outstanding mathematicians of his time. This goal was accomplished in due course, and it lent credibility to his ideas and results about constructivity and "intuitionistic" mathematics.

In retrospect we see that his standing as one of the world's leading mathematicians made it difficult to ignore his development of a revolutionary new thread in the long history of mathematics. His work has also had a major impact on areas of computer science. As we have mentioned, the Nuprl proof assistant now implements a fully intuitionistic mathematics including the intuitionistic notion of type (species and spreads), intuitionistic logic, free choice sequences, Bar Induction, the Continuity Principle, and Brouwer's way of interpreting classical logical operators. His elegant constructive proof of $\neg\neg(A \vee \neg A)$ is something many people find fascinating.

A natural question is whether there are constructive proofs of Brouwer's fixed point theorems. So far we do not have constructive or intuitionistic proofs of general fixed point theorems. On the other hand, as with the Intermediate Value Theorem, it is known how to prove ϵ versions, that is approximations of the fixed point. We look at one of these results and note its similarity to the weak IVT.[2]

One of the best ways to understand the value of constructive and intuitionistic mathematics is to look at concrete theorems of interest such as the fixed point theorems. This approach is concrete and meaningful because it is grounded in specific mathematical questions and the quality of the answers we can give. Nowadays the importance of algorithms and computational solutions are easy to motivate because interesting and important algorithms are increasingly critical in modern life. They automatically park cars and land airplanes. They explain how animals reproduce and how to bring spaceships back to earth when mechanical systems malfunction. We know that robots are following algorithms when they do smart things or bad things. There are books about modern warfare that make no sense unless the reader understands how malicious algorithms can be injected into industrial processes critical to bomb making. The whole idea of *cyber crime and cyber warfare* is now something that informed citizens need to understand just to stay safe and to know how to vote on matters of safety in a world increasingly dependent on algorithms.

There are many popular books on this topic of algorithms in modern life [?, ?, 11, 12, 13, 18, 10]. One of the very early books along these lines is from one of the great mathematicians of all time [?].

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[2]Our colleagues have asked us many times why we don't believe $(A \vee \neg A)$. For years we gave an explanation, but at one point we asked why they believed it. Andrew Myers said, because "it can't possibly be false," and he immediately gave the intuitive proof. Indeed it is Brouwer's proof. We said "yes, we fully agree." There is a lesson in that exchange, beyond the fact that Andrew is very smart.

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