CS/Math-4860fa-2019-Lecture 7 Foundational Issues and iFOL

September 23, 2019

Abstract

We have now studied classical first-order logic from Raymond Smullyan's classical short text book *First-Order Logic.* We have commented that set theory can be axiomatized first-order logic. Many mathematicians believe that we have now covered the standard classical foundations of mathematics. Some believe that this is all of the logic a person needs.

However, from the very first week of the course, we read and discussed the views of one great mathematician, L.E.J. Brouwer. He thought that logic "got in the way" of real mathematics, which is based on intuition not logic. We will see in the next few lectures that we can formulate an intuitionistic first-order logic, iFOL, that accounts for the core of intuitionistic mathematics. In due course we will see that this logic can express most of classical mathematics. However, we will see that set theory is not the most appropriate basis. Instead, *type theory is more appropriate* for expressing computational foundations.

1 Lecture 1 Summary

2 Krönecker

Krönecker was a distinguished German mathematician known for his work in algebraic and analytic number theory. He is also known for being opposed to using the concept of a completed infinite set in mathematical reasoning as Cantor and others were starting to do. He agreed with Gauss that the idea of an infinite set was just a manner of speaking about collections such as the natural numbers, 0;1;2;3; ... which can be continued without end because given any number n, we can construct a larger number by adding one to it. He was in particular opposed to Cantor's work on set theory, but he also disagreed with methods of proof that did not produce concrete answers. He strongly favored computational methods and explicit constructions.

Krönecker is mentioned in Bell's *Men of Mathematics* as being "viciously opposed" to Cantor and others who proved results about infinite sets as if they were completed totalities, but this appears to be a considerable exaggeration. What is true is that in 1886 he made an after dinner speech in which he said "God made the integers, all else is the work of man" as a way of expressing his interest in constructions. Here is the German for what he said:

"Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk." We will look next at a mathematician who did openly attack non-computational methods and who was in contention for being the leading mathematician of his day.

3 Brouwer

In his 1907 doctoral dissertation, *On the Foundations of Mathematics* [1], Brouwer provided a meaning for mathematical statements based on mental constructions. These constructions are intuitively known to be effective for basic mathematical tasks. This philosophical stance is known as *intuitionism*, and Brouwer was interested in building mathematics according to this philosophy. Brouwer believed that the crisis in the foundations of analysis was due to mathematicians not understanding the full extent of constructive methods. In particular he believed that our mental constructions are the proper justification for logic and that they do not justify even full first-order logic.

One of the fundamental logical laws that is not justified according to Brouwer is the *law of excluded middle*, P or not P, for any proposition P. For Brouwer, to assert P is to know how to prove it, and he could imagine propositions which we could never prove or disprove. Brouwer believed that logic was the study of a particular subset of abstract constructions but that it played no special role in the foundations of mathematics, it was simply a form of mathematics.

Brouwer believed that our mathematical intuitions concern two basic aspects of mathematics, the discrete and the continuous. Our intuitions about discreteness and counting discrete objects give rise to number theory, and our intuitions about time give rise to the notion of continuity and real numbers. He believed that mathematicians had not recognized the rich constructions need for understanding real numbers, including for computing with real numbers. He was determined to study these constructions and thus settle the disputed issues in analysis and in the theory or real numbers. But first he decided he should establish himself as the best mathematician in the world.

He proceeded to develop point set topology and proved his famous and widely used *fixed point theorem*. He attracted several converts and fellow travelers, and when one of the most promising young mathematicians, Weyl, became a follower of Brouwer, another contender for "world's best mathematician", David Hilbert, entered the fray in a "frog and mouse war" with Brouwer. Hilbert is famous for saying that "we can know and must know" the truth or falsity of any mathematical statement. He is also famous for *Hilbert spaces* and many more fundamental concepts.

References

[1] L.E.J. Brouwer. *Over de grondslagen der wiskunde (On the foundations of mathematics)*. PhD thesis, Amsterdam and Leipzig, 1907.