§ 1. Analytic Proofs of the Compactness Theorem

We can pose the question in another way. Consider $S$ arranged in some denumerable sequence $X_1, X_2, \ldots, X_n, \ldots$. To say that every finite subset of $S$ is satisfiable is to say nothing more nor less than that for each $n$, the set $\{X_1, \ldots, X_n\}$ is satisfiable. For clearly, if all finite subsets of $S$ are satisfiable, then for any $n$, the finite set $\{X_1, \ldots, X_n\}$ is satisfiable. Conversely, suppose that for each $n$, the set $\{X_1, \ldots, X_n\}$ is satisfiable. Then any finite subset $S_0$ of $S$ is a subset of $\{X_1, \ldots, X_n\}$ for some $n$, and hence is satisfiable.

We can thus look at the question as follows. Suppose that there is some Boolean valuation $v_1$ in which $X_1$ is true, and that there is a Boolean valuation $v_2$ (but not necessarily the same as $v_1$!) in which $X_1$ and $X_2$ are both true, and for each $n$ there is a Boolean valuation $v_n$ in which the first $n$ terms are true. Does there necessarily exist one Boolean valuation $v$ in which all the $X_i$ are simultaneously true?

We shall call a set $S$ consistent if every finite subset of $S$ is satisfiable (this is equivalent to saying that no formal contradiction can be derived from $S$ by the tableau method, i.e. there exists no finite number of elements $X_1, \ldots, X_n$ such that there is a closed tableau for $T(X_1 \land X_2 \land \cdots \land X_n)$). So the compactness question rephrased is whether a consistent infinite set is necessarily satisfiable—in other words, if it is impossible to derive a formal contradiction from $S$, is there necessarily an interpretation in which every element of $S$ is true? We shall return to this problem shortly.

König's Lemma. We first wish to consider a related problem, not about formulas of propositional logic, but about trees. Suppose $T$ is a tree in which every branch is finite. Does it necessarily follow that $T$ contains a branch of maximal length? Stated otherwise, if there exists no finite branch of maximal length, must $T$ necessarily contain an infinite branch? Stated yet another way, if for every finite $n$, there is at least one point of level $n$, does $T$ necessarily contain an infinite branch?

The answer for trees in general is "no", but for finitely generated trees (i.e. for trees in which each point has only finitely many successors) the answer is "yes".

A simple example of an infinitely generated tree for which the answer is "no" is the following:

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    a_0
   /    \
  a_1   a_2
 /     / \
 a_22  a_32  ...
 /     /     / \
 a_33  a_3  a_4
    /     /     /  \
   a_42 a_43 a_4  ...
    /  \
   a_{n2} a_{n3} a_n
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In this tree, $a_0$ branches to infinitely many points $a_1, a_2, \ldots, a_n, \ldots$. Every branch of this tree is finite, yet for each positive integer $n$, there is a branch of length greater than $n$ (the branch going through $a_{n+1}$).

Now König's lemma is to the effect that for a finitely generated tree, if for each $n$ there is at least one point of level $n$, then the tree must contain at least one infinite branch. Let us first observe that for a finitely generated tree $T$, the statement that for every $n$ there is at least one point of level $n$ is equivalent to the statement that $T$ has infinitely many points. (The first statement obviously implies the second, and conversely if $T$ has infinitely many points, they must be scattered at infinitely many levels, since each level of a finitely generated tree contains only finitely many points.) We thus shall prove König's lemma in the following equivalent form.

**König's Lemma.** Every finitely generated tree $T$ with infinitely many points must contain at least one infinite branch.

Many proofs of König's lemma exist in the literature. We shall give König's original proof, which is perhaps the shortest.

**Proof.** Call a point of $T$ good if it has infinitely many descendents (i.e. if it dominates infinitely many points) and bad if it has only finitely many descendents. By hypothesis there are infinitely many points on $T$, and they are all dominated by the origin; hence the origin is good.

We next observe that if all successors of a point are bad, then the point must be bad (since by hypothesis it has only finitely many successors). Thus a good point must have at least one good successor. Thus the origin $a_0$ has a good successor $a_1$, which in turn has a good successor $a_2$, which in turn has a good successor $a_3$, etc. In this way we generate an infinite branch $(a_0, a_1, a_2, \ldots, a_n, \ldots)$.

**Remarks.** (1) For unordered trees, the axiom of choice is needed in the above proof, since at each stage we must choose a good successor. For ordered trees, the axiom of choice is not needed, since at each stage we can always choose the leftmost good successor.

(2) The crucial place where we used the hypothesis that $T$ is finitely generated was in the statement that a good point necessarily has a good successor. This statement fails in general for infinitely generated trees—e.g. in the counterexample we considered a while back, the origin $a_0$ is good, but each of its successors $a_1, a_2, \ldots, a_n, \ldots$ is bad.

*The Compactness Problem Resumed.* The answer to the compactness question is "yes"—i.e. if all finite subsets of $S$ are satisfiable, then $S$ is satisfiable.

This theorem shall be basic for our subsequent study of first order logic, and it will be profitable to consider several proofs of this result.