Spring 2018 CS4860 Applied Logic – Sample Questions

Abstract

This document presents sample questions indicative of the kind that will be on the final exam.

1 Using the Refinement Logic Rules for FOL

In Lecture 5 the rules for First-Order Logic (FOL) are given in the refinement style. They include the classical **law of excluded middle** as the last rule. We use the justification **magic**(A) as evidence for $(A \vee \neg A)$. As we have mentioned, FOL assumes that the domain of discourse, D is non-empty. We can express this in various ways. If the basic logic included an equality relation on the domain, then we could include the axiom $\exists x : D.(x = x)$. Since we have the logical constant $False$ we could also state the existence axiom as $\exists x : D.\neg\, False$.

In Lecture 11 we compared the refinement style rules to the Smullyan's tableaux rules [2].

All of the proofs in the exam questions should use these refinement style rules rather than the tableaux rules. This is because the refinement rules provide functional programs as realizers even for classical FOL. Fitting's rules [1] for iFOL do not provide realizers.

1. What is the conceptually simplest way to know that a formula of PC is provable? How can we explain the method of the Tableaux proof system for PC in terms of that simple proof system? What is the major drawback of that system?

2. If we add the rule $P \vee \neg P$ to iPC with the realizer $magic(P)$, to the refinement rules for iPC, then we have the classical propositional calculus, PC. Using *magic* we can give refinement proofs for PC formulas that are tautologies. This explanation of how we know formulas to be "true" is quite different from the truth functional semantics. Give an example of proofs done each way for the formula

$$(P \Rightarrow (Q \Rightarrow R)) \Rightarrow ((P \Rightarrow Q) \Rightarrow (P \Rightarrow R)).$$

3. Prove the following statement in iPC if it is constructively true, otherwise explain why it is not constructively true:

$$((A \& B) \Rightarrow (C \& D)) \Rightarrow (\neg D) \Rightarrow (\neg A \vee \neg B.)$$

4. In FOL we have the simple theorem $(A \Rightarrow B) \Leftrightarrow (\neg A \vee B)$ Give the classical refinement logic proof of this theorem. Explain which implication is constructively valid, and show the realizer. Explain informally why there is no realizer for the other direction of the implication.

5. Prove $\exists x : D.(P(x) \Rightarrow C) \Rightarrow (\forall x : D \Rightarrow C)$.

6. Classically we also know $\exists x : D.(P(x) \Rightarrow C) \Leftrightarrow (\forall x : D \Rightarrow C)$. Prove this using magic, i.e. the Law of Excluded Middle (LEM).

7. Explain informally why the previous formula is not constructively true. Also explain why it "makes sense" classically.

8. Give an informal intuitive argument that *FOL with refinement rules is complete* given what we know from Smyllyan about the Tableaux rules for FOL.

9. Some people claim that classical real analysis is not axiomatizable in FOL. There is a sense in which this is clearly wrong, say why. Speculate on what people really mean when they claim this.

2 Real Analysis

We examined the notion of the constructive real numbers as defined by Bishop and Bridges. Here are some simple questions on that topic.

1. Sketch the argument that if real numbers satisfy $a < b$, then we can find a real strictly between them.

2. Using what you know from the theory of computing, and assuming that real numbers are given by Turing machines, why can't we decide the equality of real numbers?

References

[1] M. Fitting. *Intuitionistic model theory and forcing.* North-Holland, Amsterdam, 1969.

[2] R. M. Smullyan. *First–Order Logic.* Springer-Verlag, New York, 1968.