# Spring 2018 CS4860 Applied Logic – Review

**Abstract**

This document reviews key topics in the course. It relates them to each other and discusses the role of constructive type theory in integrating them into a *coherent foundation for constructive mathematics and computer science*. It also includes a few sample questions similar to what might be on the exam.

The document mentions the fact that at Cornell we have provided the *first implementation* of Brouwer's *intuitionistic type theory* in our Nuprl proof assistant. We formalized and implemented the fundamental principles of Brouwer's intuitionistic mathematics, *bar induction* and the *continuity principle*. These required that we implement the notion of *free choice sequences*, briefly discussed in the course.

We did not devote course time to explore details of the fundamental new features of *intuitionistic mathematics*. It is interesting to know that these principles have been implemented and are now being used in real analysis. As far as we know, Nuprl is the only proof assistant that implements these principles, providing the first implementation of fully intuitionistic mathematics. It is highly likely that other proof assistants will implement intuitionistic principles in due course. In this summary document we mention at most one or two new result in the area, discovered and published while the course was progressing. For instance, we have shown that *Markov's Principle is false* in the Cornell implementation of intuitionistic type theory. This important was discovered and confirmed as this course was being taught. You were among the first to know about it when it was mentioned in the final lecture.

## 1 Review of Important Logics and Topics

This course on Applied Logic covered the major topics in both pure logic, an ancient subject, and in applied logic, a modern subject strongly tied to formal methods and computer science. We used Raymond Smullyan's book *First-Order Logic* [7] as a starting point and then added material from other sources and wrote new notes on topics related to recent Cornell research in this very active area. Raymond Smullyan has written other very interesting and lively books on logic [9, 8, 10, 11]. His student Melvin Fitting also wrote two excellent books and edited others [3, 4, 5, 2].[1]

In the final lecture we mentioned some new results in applied logic and type theory including the discovery that Markov's Principle is false in fully intuitionistic type theory [1]. It is important in this class to know the statement of *Markov's Principle*.

# 2 Main Topics

Here is a list of the lectures and the most key concepts in each one.

1. Introductory Lectures 1 and 2: Logic is an ancient subject. It is good to know roughly how old and who the earliest eminent logicians were and who are some of the major contributors since then. Lecture 2 provides a brief historical overview. Students should be familiar with the eighteen names mentioned and what they contributed.

2. Propositional logic: Lectures 3 and 4 defined propositional logic from Smullyan using truth functional semantics. It is important to know what **consistency**, **completeness**, and **decidability** mean for this simple logic and more generally.

3. Lectures 5 and 6: We discussed the meaning of theorems in mathematics of the form, If the generalized continuum hypothesis (GCH) is true, then P = NP. Here we have two precise mathematical statements which are both open problems. Thus we don't know their truth values, and we might never know them; yet there is a provable relationship between them. How can that be? We discussed the notion of **evidence** for a proposition and how that is used in proving assertions such as GCH implies (P = NP), a proven result. We then gave rules of evidence for the logic **iPC**, the i**intuitionistic Propositional Calculus**.

4. Lectures 7 and 8: We illustrated the tableaux style of proof for PC and discussed Tableaux style rules for **iPC**.

5. Lectures 9 and 10: We introduced **First-Order Logic, FOL** and **intuitionistic First-Order Logic iFOL**. These are the most important logics in the course. In mathematics it is consider sufficient to FOL, and students should know why that is the case.

[1]Smullyan was also a magician, and you can see elements of this in his books.

6. Lecture 11 and 12: We introduced Kleene's axioms for number theory, in particular for **Peano Arithmetic (PA)** and *intuitionistic Arithmetic* called **Heyting Arithmetic (HA)**. We looked at L.E.J. Brouwer's thoughts on intuitionistic truth.

7. Lectures 13 and 14: We studied the **completeness proof for FOL** and discussed the role of Konig's Lemma. This is a major result in the course.

8. Lecture 15: Embedding FOL into iFOL. We examined a translation of classical logical operators into constructive equivalents. For example, $(A \vee B)^o\ is\ \neg(\neg A^o \& \neg B^o)$.

9. Lectures 16 and 17: Completeness of FOL is discussed further, and we mentioned a constructive proof by J.L.Krivine [6]. **This topic will not be covered on the final exam**. We did not discuss Church's Theorem that there is no decision procedure for provability in FOL. This is a deep and famous theorem, so we should all recall the statement of the theorem. There is good information on the web.

10. Lectures 18, 19, and 20: We discussed the **real numbers** $\mathbb{R}$ **the nonstandard real numbers**, $\mathbb{R}^\star$, also called the hyper-reals. This topic will also not be covered on the final exam. In the course of this discussion we looked at axioms for the real numbers. If we want to axiomatize the real numbers without developing them in set theory, the question about whether there is a first-order axiomatization is more delicate. We did not cover this issue very deeply

11. Lecture 21 and 22: Bishop's account of the **constructive real numbers**. It will be important to know the definition of these numbers and the definition of equality and inequality. It is also important to understand the addition of real numbers and the concept of one number being greater than another.

12. Lectures 23 and 24: **Computational type theory**. It is important to know the definition of a type and to understand how type theory integrates our accounts of first-order logic, Heyting Arithmetic, and Constructive Reals.

13. Lecture 25 was a guest lecture on homotopy theory by Siva Somayyajula, the course TA. This will not be on the final exam.

14. Lecture 26 was a guest lecture by Ariel Kellison on *Euclidean Geometry.* It is good to know the statement and importance of Euclid's Proposition 2 as well as the constructive issue concerning it.

15. Lecture 27 was a review of the course and a discussion of the integrating role of type theory.

## 3 Sample Questions

Having this background in logic might help you untangle misleading statements you can find on the web and in survey articles. We just found something lately from people trying to axiomatize relativity theory. They claim that FOL cannot model the real numbers R, so it cannot be used for this formalization. What is misleading about that claim? What might they want to say?

- Prove the following statement in iPC if it is constructively true, otherwise explain why it is not constructively true:

$$((A\&B) \Rightarrow (C\&D)) \Rightarrow \neg D \Rightarrow (\neg A \vee \neg B).$$

- Is Markov's Principle classically true? If so prove it in FOL, if not explain why.

- It is convenient to add a **sequencing rule**, also called the **cut rule**, such as the following.

  $H \vdash G \;\; by \;\; Seq(S)$
  $H, S \vdash G$
  $H \vdash S$

  Explain why the rule is valid.

  There is an important result in proof theory that says that any proof using cut can be done without it. What is a simple way to understand this result? In other words, why do we know that this should be true in the normal way mathematics is done informally. What is a standard way to accomplish what this rule accomplishes if we were writing an informal explanation of why $G$ is true given the hypotheses in $H$?

## References


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[5] Melvin Fitting and Brian Rayman, editors. *Raymond Smullyan on Self Reference.* Springer, 2017.

[6] Jean-Louis Krivine. Une preuve formelle et intuitionniste du thorme de compltude de la logique classique. *Bulletin of Symbolic Logic*, 2:405–421, 12 1996.

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[10] Raymond M. Smullyan. *Gödel's Incompleteness Theorems.* Oxford University Press, New York, 1992.

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