CHAPTER 1

PROPOSITIONAL INTUITIONISTIC LOGIC

SEMANTICS

§ 1. Formulas

We begin with a denumberable set of propositional variables A, B, C,..., three binary connectives $\wedge$, $\vee$, $\supset$, and one unary connective $\sim$, together with left and right parentheses (,). We shall informally use square and curly brackets [,], {, } for parentheses, to make reading simpler. The notion of *well formed formula*, or simply *formula*, is given recursively by the following rules:

F0. If A is a propositional variable, A is a formula.

F1. If X is a formula, so is $\sim X$.

F2, 3, 4. If X and Y are formulas, so are $(X \wedge Y)$, $(X \vee Y)$, $(X \supset Y)$.

Remark 1.1: A propositional variable will sometimes be called an *atomic formula.*

It can be shown that the formation of a formula is unique. That is, for any given formula X, one and only one of the following can hold:

(1). X is A for some propositional variable A.

(2). There is a unique formula Y such that X is $\sim Y$.

(3). There is a unique pair of formulas Y and Z and a unique binary connective b ($\wedge$, $\vee$ or $\supset$) such that X is $(Y \mathrm{b} Z)$.

We make use of this uniqueness of decomposition but do not prove it here.

We shall omit writing outer parentheses in a formula when no confusion can result. Until otherwise stated, we shall use A, B and C for propositional variables, and X, Y and Z to represent any formula.

The notion of *immediate subformula* is given by the following rules:

I0. A has no immediate subformula.

I1. $\sim X$ has exactly one immediate subformula: X.

I2, 3, 4. $(X \wedge Y)$, $(X \vee Y)$, $(X \supset Y)$ each has exactly two immediate subformulas: X and Y.

The notion of *subformula* is defined as follows:

S0. X is a subformula of X.

S1. If X is an immediate subformula of Y, then X is a subformula of Y.

S2. If X is a subformula of Y and Y is a subformula of Z, then X is a subformula of Z.

By the *degree* of a formula is meant the number of occurrences of logical connectives ($\sim, \wedge, \vee, \supset$) in the formula.

§ 2. Models and validity

By a (*propositional intuitionistic*) *model* we mean an ordered triple $\langle \mathscr{G}, \mathscr{R}, \models \rangle$, where $\mathscr{G}$ is a non-empty set, $\mathscr{R}$ is a transitive, reflexive relation on $\mathscr{G}$, and $\models$ (conveniently read "forces") is a relation between elements of $\mathscr{G}$ and formulas, satisfying the following conditions:

For any $\Gamma \in \mathscr{G}$

P0. if $\Gamma \models A$ and $\Gamma \mathscr{R} \Delta$ then $\Delta \models A$ (recall A is atomic).

P1. $\Gamma \models (X \wedge Y)$ iff $\Gamma \models X$ and $\Gamma \models Y$.

P2. $\Gamma \models (X \vee Y)$ iff $\Gamma \models X$ or $\Gamma \models Y$.

P3. $\Gamma \models \sim X$ iff for all $\Delta \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Delta$, $\Delta \not\models X$.

P4. $\Gamma \models (X \supset Y)$ iff for all $\Delta \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Delta$, if $\Delta \models X$, then $\Delta \models Y$.

Remark 2.1: For $\Gamma \in \mathscr{G}$, by Γ^* we shall mean any $\Delta \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Delta$. Thus "for all Γ^*, $\varphi(\Gamma^*)$" shall mean "for all $\Delta \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Delta$, $\varphi(\Delta)$"; and "there is a Γ^* such that $\varphi(\Gamma^*)$" shall mean "there is a $\Delta \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Delta$ and $\varphi(\Delta)$". Thus P3 and P4 can be written more simply as:

P3. $\Gamma \models \sim X$ iff for all Γ^*, $\Gamma^* \not\models X$

P4. $\Gamma \models (X \supset Y)$ iff for all Γ^*, if $\Gamma^* \models X$, then $\Gamma^* \models Y$.

A particular formula X is called *valid in the model* $\langle \mathscr{G}, \mathscr{R}, \models \rangle$ if for all $\Gamma \in \mathscr{G}$, $\Gamma \models X$. X is called *valid* if X is valid in all models. We will show later that the collection of all valid formulas coincides with the usual collection of propositional intuitionistic logic theorems.

When it is necessary to distinguish between validity in this sense and the more usual notion, we shall refer to the validity defined above as intuitionistic validity, and the usual notion an classical validity. This notion of an intuitionistic model is due to Saul Kripke, and is presented, in different notation, in [13]. See also [18]. Examples of models will be found in section 5, chapter 2.

§ 3. Motivation

Let $\langle \mathscr{G}, \mathscr{R}, \models \rangle$ be a model. $\mathscr{G}$ is intended to be a collection of possible universes, or more properly, states of knowledge. Thus a particular Γ in $\mathscr{G}$ may be considered as a collection of (physical) facts known at a particular time. The relation $\mathscr{R}$ represents (possible) time succession. That is, given two states of knowledge Γ and Δ of $\mathscr{G}$, to say $\Gamma \mathscr{R} \Delta$ is to say: if we now know Γ, it is possible that later we will know Δ. Finally, to say $\Gamma \models X$ is to say: knowing Γ, we know X, or: from the collection of facts Γ, we may deduce the truth of X.

Under this interpretation condition P3 of the last section, for example, may be interpreted as follows: from the facts Γ we may conclude $\sim X$ if and only if from no possible additional facts can we conclude X.

We might remark that under this interpretation it would seem reasonable that if $\Gamma \models X$ and $\Gamma \mathscr{R} \Delta$ then $\Delta \models X$, that is, if from a certain amount of information we can deduce X, given additional information, we still can deduce X, or if at some time we know X is true, at any later time we still know X is true. We have required that this holds only for the case that X is atomic, but the other cases follow.

For other interpretations of this modeling, see the original paper [13]. For a different but closely related model theory in terms of forcing see [5].

§ 4. Some properties of models

***Lemma* 4.1:** Let $\langle \mathscr{G}, \mathscr{R}, \models \rangle$ and $\langle \mathscr{G}, \mathscr{R}, \models' \rangle$ be two models such that for any atomic formula A and any $\Gamma \in \mathscr{G}$, $\Gamma \models A$ iff $\Gamma \models' A$. Then $\models$ and $\models'$ are identical.

Proof: We must show that for any formula X,

$$\Gamma \vDash X \Leftrightarrow \Gamma \vDash' X .$$

This is done by induction on the degree of X and is straightforward. We present one case as an example.

Suppose X is $\sim Y$ and the result is known for all formulas of degree less than that of X (in particular for Y). We show it for X:

$$\begin{aligned} \Gamma \vDash X &\Leftrightarrow \Gamma \vDash \sim Y \quad \text{(by definition)} \\ &\Leftrightarrow (\forall \Gamma^*)(\Gamma^* \nvDash Y) \quad \text{(by hypothesis)} \\ &\Leftrightarrow (\forall \Gamma^*)(\Gamma^* \nvDash' Y) \quad \text{(by definition)} \\ &\Leftrightarrow \Gamma \vDash' \sim Y \\ &\Leftrightarrow \Gamma \vDash' X . \end{aligned}$$

***Lemma* 4.2:** Let $\mathscr{G}$ be a non-empty set and $\mathscr{R}$ be a transitive, reflexive relation on $\mathscr{G}$. Suppose $\vDash$ is a relation between elements of $\mathscr{G}$ and *atomic* formulas. Then $\vDash$ can be extended to a relation $\vDash'$ between elements of $\mathscr{G}$ and *all* formulas in such a way that $\langle \mathscr{G}, \mathscr{R}, \vDash' \rangle$ is a model.

Proof: We define $\vDash'$ as follows:

(0). if $\Gamma \vDash A$ then $\Gamma^* \vDash' A$,

(1). $\Gamma \vDash' (X \wedge Y)$ if $\Gamma \vDash' X$ and $\Gamma \vDash' Y$,

(2). $\Gamma \vDash' (X \vee Y)$ if $\Gamma \vDash' X$ or $\Gamma \vDash' Y$,

(3). $\Gamma \vDash' \sim X$ if for all Γ^*, $\Gamma^* \nvDash' X$,

(4). $\Gamma \vDash' (X \supset Y)$ if for all Γ^*, if $\Gamma^* \vDash' X$, then $\Gamma^* \vDash' Y$.

This is an inductive definition, the induction being on the degree of the formula. It is straightforward to show that $\langle \mathscr{G}, \mathscr{R}, \vDash' \rangle$ is a model.

From lemmas 4.1 and 4.2 we immediately have

***Theorem* 4.3:** Let $\mathscr{G}$ be a non-empty set and $\mathscr{R}$ be a transitive, reflexive relation on $\mathscr{G}$. Suppose $\vDash$ is a relation between elements of $\mathscr{G}$ and atomic formulas. Then $\vDash$ can be extended in one and only one way to a relation, also denoted by $\vDash$, between elements of $\mathscr{G}$ and formulas, such that $\langle \mathscr{G}, \mathscr{R}, \vDash \rangle$ is a model.

***Theorem* 4.4:** Let $\langle \mathscr{G}, \mathscr{R}, \vDash \rangle$ be a model, X a formula and $\Gamma, \Delta \in \mathscr{G}$. If $\Gamma \vDash X$ and $\Gamma \mathscr{R} \Delta$, then $\Delta \vDash X$.

Proof: A straightforward induction on the degree of X (it is known already for X atomic). For example, suppose the result is known for X, and $\Gamma \vDash \sim X$. By definition, for all Γ^*, $\Gamma^* \nvDash X$. But $\Gamma \mathscr{R} \Delta$ and $\mathscr{R}$ is transitive so any $\mathscr{R}$-successor of Δ is an $\mathscr{R}$-successor of Γ. Hence for all Δ^*, $\Delta^* \nvDash X$, so $\Delta \vDash \sim X$. The other cases are similar.

§ 5. Algebraic models

In addition to the Kripke intuitionistic semantics presented above, there is an older algebraic semantics: that of pseudo-boolean algebras. In this section we state the algebraic semantics, and in the next we prove its equivalence with Kripke's semantics. A thorough treatment of pseudo-boolean algebras may be found in [16].

Definition 5.1: A *pseudo-boolean algebra* (PBA) is a pair $\langle \mathscr{B}, \leqslant \rangle$ where $\mathscr{B}$ is a non-empty set and $\leqslant$ is a partial ordering relation on $\mathscr{B}$ such that for any two elements a and b of $\mathscr{B}$:

(1). the least upper bound $(a \cup b)$ exists.
(2). the greatest lower bound $(a \cap b)$ exists.
(3). the pseudo complement of a relative to b $(a \Rightarrow b)$, defined to be the largest $x \in \mathscr{B}$ such that $a \cap x \leqslant b$, exists.
(4). a least element $\wedge$ exists.

Remark 5.2: In the context $\Rightarrow$ is a mathematical symbol, not a metamathematical one.

Let $-a$ be $a \Rightarrow \wedge$ and $\vee$ be $- \wedge$.

Definition 5.3: h is called a *homomorphism* (from the set W of formulas to the PBA $\langle \mathscr{B}, \leqslant \rangle$) if $h: W \to \mathscr{B}$ and

(1). $h(X \wedge Y) = h(X) \cap h(Y)$,
(2). $h(X \vee Y) = h(X) \cup h(Y)$,
(3). $h(\sim X) \; = -h(X)$,
(4). $h(X \supset Y) = h(X) \Rightarrow h(Y)$.

If $\langle \mathscr{B}, \leqslant \rangle$ is a PBA and h is a homomorphism, the triple $\langle \mathscr{B}, \leqslant, h \rangle$ is called an *(algebraic) model* for the set of formulas W. If X is a formula, X is called *(algebraically) valid in the model* $\langle \mathscr{B}, \leqslant, h \rangle$ if $h(X) = \vee$. X is called *(algebraically) valid* if X is valid in every model.

A proof may be found in [16] that the collection of all algebraically valid formulas coincides with the usual collection of intuitionistic theorems.

§ 6. Equivalence of algebraic and Kripke validity

First let us suppose we have a Kripke model $\langle \mathscr{G}, \mathscr{R}, \models \rangle$ (we will not use the name "Kripke model" beyond this section). We will define an algebraic model $\langle \mathscr{B}, \leqslant, h \rangle$ such that for any formula X

$$h(X) = \mathsf{v} \quad \text{iff} \quad \text{for all} \quad \Gamma \in \mathscr{G}, \; \Gamma \vDash X.$$

Remark 6.1: The following proof is based on exercise LXXXVI of [2].

If $b \subseteq \mathscr{G}$, we call b *$\mathscr{R}$-closed* if whenever $\Gamma \in b$ and $\Gamma \mathscr{R} \Delta$, then $\Delta \in b$.

We take for $\mathscr{B}$ the collection of all $\mathscr{R}$-closed subsets of $\mathscr{G}$. For the ordering relation $\leqslant$ we take set inclusion $\subseteq$. Finally we define h by

$$h(X) = \{\Gamma \in \mathscr{G} \mid \Gamma \vDash X\}.$$

It is fairly straightforward to show that $\langle \mathscr{B}, \leqslant \rangle$ is a PBA. Of the four required properties, the first two are left to the reader. We now show:

If $a, b \in \mathscr{B}$, there is a largest $x \in \mathscr{B}$ such that $a \cap x \leqslant b$.

We first note that the operations $\cup$ and $\cap$ are just the ordinary union and intersection. Now let p be the largest $\mathscr{R}$-closed subset of $(\mathscr{G} \dot{-} a) \cup b$ (where by $\dot{-}$ we mean ordinary set complementation). We will show that for all $x \in \mathscr{B}$

$$x \leqslant p \quad \text{iff} \quad a \cap x \leqslant b,$$

which suffices.

Suppose $x \leqslant p$. Then

$$\begin{aligned} &x \subseteq (\mathscr{G} \dot{-} a) \cup b, \\ &a \cap x \subseteq a \cap [(\mathscr{G} \dot{-} a) \cup b], \\ &a \cap x \subseteq a \cap b, \\ &a \cap x \subseteq b, \\ &a \cap x \leqslant b. \end{aligned}$$

Conversely suppose $a \cap x \leqslant b$. Then

$$\begin{aligned} (a \cap x) \cup (x \dot{-} a) &\subseteq b \cup (x \dot{-} a), \\ x &\subseteq b \cup (x \dot{-} a), \\ x &\subseteq b \cup (\mathscr{G} \dot{-} a), \end{aligned}$$

but $x \in \mathscr{B}$, so x is $\mathscr{R}$-closed. Hence

$$\begin{aligned} &x \subseteq p, \\ &x \leqslant p. \end{aligned}$$

The reader may verify that $\emptyset \in \mathscr{B}$ and is a least element.

Next we remark that h is a homomorphism. We demonstrate only one of the four cases, case (4). Thus we must show that $h(X \supset Y)$ is the largest

$x \in \mathscr{B}$ such that

$$h(X) \cap x \leqslant h(Y).$$

First we show

$$h(X) \cap h(X \supset Y) \leqslant h(Y),$$

that is

$$\{\Gamma \mid \Gamma \models X\} \cap \{\Gamma \mid \Gamma \models X \supset Y\} \subseteq \{\Gamma \mid \Gamma \models Y\}.$$

But it is clear from the definition that

$$\text{if } \Gamma \models X \text{ and } \Gamma \models X \supset Y, \text{ then } \Gamma \models Y.$$

Next suppose there is some $b \in \mathscr{B}$ such that $h(X) \cap b \leqslant h(Y)$ but $h(X \supset Y) < b$. Then there must be some $\Gamma \in \mathscr{G}$ such that $\Gamma \in b$ but $\Gamma \notin h(X \supset Y)$, i.e. $\Gamma \not\models X \supset Y$. Since $\Gamma \not\models X \supset Y$, there must be some Γ^* such that $\Gamma^* \models X$ but $\Gamma^* \not\models Y$. Since b is $\mathscr{R}$-closed, $\Gamma^* \in b$. But also $\Gamma^* \in h(X)$, so $\Gamma^* \in h(X) \cap b$, and so by assumption $\Gamma^* \in h(Y)$, that is $\Gamma^* \models Y$, a contradiction. Thus $h(X \supset Y)$ is largest.

Thus $\langle \mathscr{B}, \leqslant, h \rangle$ is an algebraic model. We leave it to the reader to verify that the unit element $\vee$ of $\mathscr{B}$ is $\mathscr{G}$ itself. Hence

$$h(X) = \vee \quad \text{iff} \quad \text{for all} \quad \Gamma \in \mathscr{G}, \ \Gamma \models X.$$

Conversely, suppose we have an algebraic model $\langle \mathscr{B}, \leqslant, h \rangle$. We will define a Kripke model $\langle \mathscr{G}, \mathscr{R}, \models \rangle$ so that for any formula X

$$h(X) = \vee \quad \text{iff} \quad \text{for all} \quad \Gamma \in \mathscr{G}, \ \Gamma \models X.$$

***Lemma* 6.2:** Let $\mathscr{F}$ be a filter in $\mathscr{B}$ and suppose $(a \Rightarrow b) \notin \mathscr{F}$. Then the filter generated by $\mathscr{F}$ and a does not contain b.

Proof: If the filter generated by $\mathscr{F}$ and a contained b, then ([16] p. 46, 8.2) for some $c \in \mathscr{F}$, $c \cap a \leqslant b$. So $c \leqslant (a \Rightarrow b)$ and hence $(a \Rightarrow b) \in \mathscr{F}$ by [16], p. 46, 8.2 again.

***Lemma* 6.3:** Let $\mathscr{F}$ be a proper filter in $\mathscr{B}$ and suppose $-a \notin \mathscr{F}$. Then the filter generated by $\mathscr{F}$ and a is also proper.

Proof: By lemma 6.2, since $-a = (a \Rightarrow \wedge)$.

***Lemma* 6.4:** Let $\mathscr{F}$ be a filter in $\mathscr{B}$ and suppose $a \notin \mathscr{F}$. Then $\mathscr{F}$ can be extended to a prime filter $\mathscr{P}$ such that $a \notin \mathscr{P}$.

Proof: (This is a slight modification of [16], p. 49, 9.2, included for completeness.) Let $\boldsymbol{O}$ be the collection of all filters in $\mathscr{B}$ not containing a. $\boldsymbol{O}$ is partially ordered by $\subseteq$. $\boldsymbol{O}$ is non-empty since $\mathscr{F} \in \boldsymbol{O}$.

Any chain in $\boldsymbol{O}$ has an upper bound since the union of any chain of filters is a filter. So by Zorn's lemma $\boldsymbol{O}$ contains a maximal element $\mathscr{P}$. Of course $a \notin \mathscr{P}$. We need only show $\mathscr{P}$ is prime.

Suppose $\mathscr{P}$ is not prime. Then for some $a_1, a_2 \in \mathscr{B}$

$$a_1 \cup a_2 \in \mathscr{P}, \quad a_1 \notin \mathscr{P}, \quad a_2 \notin \mathscr{P}.$$

Let $\mathscr{S}_1$ be the filter generated by $\mathscr{P}$ and a_1, and $\mathscr{S}_2$ be the filter generated by $\mathscr{P}$ and a_2.

Suppose $a \in \mathscr{S}_1$ and $a \in \mathscr{S}_2$. Then [16, p. 46, 8.2] for some $c_1, c_2 \in \mathscr{P}$, $a_1 \cap c_1 \leqslant a$ and $a_2 \cap c_2 \leqslant a$. So for $c = c_1 \cap c_2$, $a_1 \cap c \leqslant a$ and $a_2 \cap c \leqslant a$, hence $(a_1 \cup a_2) \cap c \leqslant a$. But $c \in \mathscr{P}$ and $(a_1 \cup a_2) \in \mathscr{P}$, so $a \in \mathscr{P}$. But $a \notin \mathscr{P}$, so either $a \notin \mathscr{S}_1$ or $a \notin \mathscr{S}_2$.

Suppose $a \notin \mathscr{S}_1$. By definition $\mathscr{S}_1 \in \boldsymbol{O}$. But $\mathscr{S}_1$ is the filter generated by $\mathscr{P}$ and a_1, hence $\mathscr{P} \subseteq \mathscr{S}_1$. So $\mathscr{P}$ is not maximal, a contradiction. Similarly if $a \notin \mathscr{S}_2$. Thus $\mathscr{P}$ is prime.

Now we proceed with the main result. Recall that we have $\langle \mathscr{B}, \leqslant, h \rangle$. Let $\mathscr{G}$ be the collection of all proper prime filters in $\mathscr{B}$. Let $\mathscr{R}$ be set inclusion $\subseteq$. For any $\Gamma \in \mathscr{G}$ and any formula X, let $\Gamma \models X$ if $h(X) \in \Gamma$.

To show the resulting structure $\langle \mathscr{G}, \mathscr{R}, \models \rangle$ is a model, we note property P0 is immediate. To show P1:

$$\begin{aligned} \Gamma \models (X \wedge Y) \quad &\text{iff} \quad h(X \wedge Y) \in \Gamma \\ &\text{iff} \quad h(X) \cap h(Y) \in \Gamma \\ &\text{iff} \quad h(X) \in \Gamma \text{ and } h(Y) \in \Gamma \\ &\text{iff} \quad \Gamma \models X \text{ and } \Gamma \models Y \end{aligned}$$

(using the facts that h is a homomorphism and Γ is a filter). Similarly we show P2 using the fact that Γ is prime. To show P3:

Suppose $\Gamma \models \sim X$. Then $h(\sim X) \in \Gamma$, so

$$\begin{aligned} &(\forall \Delta \in \mathscr{G})(\Gamma \subseteq \Delta \text{ implies } h(\sim X) \in \Delta), \\ &(\forall \Delta \in \mathscr{G})(\Gamma \subseteq \Delta \text{ implies } h(X) \notin \Delta), \\ &(\forall \Delta \in \mathscr{G})(\Gamma \mathscr{R} \Delta \text{ implies } \Delta \not\models X), \end{aligned}$$

i.e. for all Γ^*, $\Gamma^* \not\models X$ (using the fact that $h(\sim X) \in \Delta$ and $h(X) \in \Delta$ imply $-h(X) \cap h(X) \in \Delta$, so $\wedge \in \Delta$ and Δ is not proper).

Suppose $\Gamma \not\models \sim X$. Then $h(\sim X) \notin \Gamma$, or $-h(X) \notin \Gamma$. By lemma 6.3 the filter generated by Γ and $h(X)$ is proper. By lemma 6.4 this filter can be extended to a proper prime filter Δ. Then $\Gamma \subseteq \Delta$ and $h(X) \in \Delta$. So $(\exists \Delta \in \mathscr{G})(\Gamma \mathscr{R} \Delta$ and $\Delta \models X)$, i.e. for some Γ^*, $\Gamma^* \models X$.

P4 is shown in the same way, but using lemma 6.2 instead of lemma 6.3. Thus $\langle \mathscr{G}, \mathscr{R}, \models \rangle$ is a model.

Finally, to establish the desired equivalence, suppose first $h(X) = \vee$. Since $\vee$ is an element of every filter, for all $\Gamma \in \mathscr{G}$, $\Gamma \models X$. Conversely suppose $h(X) \neq \vee$. But $\{\vee\}$ is a filter and $h(X) \notin \{\vee\}$. By lemma 6.4 we can extend $\{\vee\}$ to a proper prime filter Γ such that $h(X) \notin \Gamma$. Thus $\Gamma \in \mathscr{G}$ and $\Gamma \not\models X$.

Thus we have shown

***Theorem* 6.5:** X is Kripke valid if and only if X is algebraically valid.