Applied Logic - CS4860-2018fa-Lecture 16 and 17

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Abstract

These two lectures examined the question of whether the decision procedures for iPC tell us whether or not a formula of the intuitionistic Propositional Calculus is *programmable*. The account in Fitting's book, *Intuitionistic model theory and forcing* [1] is not constructive. On the other hand, due to a suggestion by his PhD advisor, Smullyan, the account is similar to the one Smullyan used in our textbook, *First-Order Logic* [4].

Judith Underwood gives a constructive proof of iPC completeness in her article posted on the course web page, *A Constructive Completeness Proof for the Intuitionistic Propositional Calculus* [5]. We showed in class how her algorithm does provide a program for the constructively true propositions and a Beth/Kripke model when the propositional formula is not programmable. She shows how the Beth/Kripke model proves that there is no program. This is an expensive procedure, and that is because *any iPC decision procedure* for provability is PSpace complete as shown by Statman.

1 Proofs of PC Completeness

The completeness the classical propositional calculus is an easy result in logic based on truth functions. Smullyan's account from his book *First-Order Logic* [4] is one of the simplest to understand. He provides a way to search systematically for a falsifying assignment of truth values to the propositional variables. If this systematic search fails, the proposition is a tautology. This is a fully constructive computational result. It provides a basis for Smullyan's proof of completeness for First-Order Logic (FOL) which we will take up in the next few lectures.

We have studied constructive proofs for iPC formulas. This has been an exercise in simple functional programming. When we find a proof of an implication, as in the homework problems such as $(A \vee B) \Rightarrow (\neg(\neg A \wedge \neg B))$ we are creating a functional program. When we are unable to find a proof by the methods taught so far, we are not sure whether the formula is intuitionistically false or whether we were just unable to discover the proof. *These results of Beth show how to demonstrate that the formula is constructively false.* They are related to similar ideas by Kripke [2]. These results have been developed further at Cornell [3, 5, 6].

References

[1] M. Fitting. *Intuitionistic model theory and forcing.* North-Holland, Amsterdam, 1969.

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[3] J. Lipton. Realizabilty and Kripke forcing. Technical Report 90–1163, Cornell University, Ithaca, NY, October 1990.

[4] R. M. Smullyan. *First–Order Logic.* Springer-Verlag, New York, 1968.

[5] Judith L. Underwood. A constructive completeness proof for the intuitionistic propositional calculus. Technical Report 90–1179, Cornell University, 1990.

[6] Judith L. Underwood. The tableau algorithm for intuitionistic propositional calculus as a constructive completeness proof. In *Proceedings of the Workshop on Theorem Proving with Analytic Tableaux, Marseille, France*, pages 245–248, 1993. Available as Technical Report MPI-I-93-213 Max-Planck-Institut für Informatik, Saarbrücken, Germany.