VII

LOGIC

7.1. The propositional calculus

The word "logic" has many different meanings. I shall not try to give a definition of intuitionistic logic, any more than I have begun this course by a definition of mathematics. Yet a preliminary remark will be useful. Our logic has only to do with mathematical propositions; the question whether it admits any applications outside mathematics does not concern us here. The letters p, q, r occur in this chapter as variables for mathematical propositions; German letters $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$ will be used as abbreviations for mathematical propositions. It is not my purpose to give a complete formal treatment of intuitionistic logic; a formal system which codifies all the logical inferences of intuitionistic mathematics known at present, is easily accessible in Kleene's book [S. C. Kleene 1952], where the reader will also find an account of the metamathematical investigations of this system. Here I shall only call your attention to some formulas which express interesting methods of reasoning and show why these methods are intuitively clear within the realm of intuitionistic mathematics.

It will be necessary to fix, as firmly as possible, the meaning of the logical connectives; I do this by giving necessary and sufficient conditions under which a complex expression can be asserted.

7.1.1. *Interpretation of the signs*

The *conjunction* $\wedge$ gives no difficulty. $\mathfrak{p} \wedge \mathfrak{q}$ can be asserted if and only if both $\mathfrak{p}$ and $\mathfrak{q}$ can be asserted.

I have already spoken of the *disjunction* $\vee$ (2.2.5, at the end). $\mathfrak{p} \vee \mathfrak{q}$ can be asserted if and only if at least one of the propositions $\mathfrak{p}$ and $\mathfrak{q}$ can be asserted.

The *negation* $\neg$ is the strong mathematical negation which we have already discussed (2.2.2). In order to give a more explicit clarification, we remember that a mathematical proposition $\mathfrak{p}$ always demands a mathematical construction with certain given properties; it can be asserted as soon as such a construction has been carried out. We say in this case that the construction *proves* the proposition $\mathfrak{p}$ and call it a *proof* of $\mathfrak{p}$. We also, for the sake of brevity, denote by $\mathfrak{p}$ any construction which is intended by the proposition $\mathfrak{p}$. Then $\neg\mathfrak{p}$ can be asserted if and only if we possess a construction which from the supposition that a construction $\mathfrak{p}$ were carried out, leads to a contradiction.

SIGN. Is it not necessary to clarify the notion of a contradiction?

INT. I think that contradiction must be taken as a primitive notion. It seems very difficult to reduce it to simpler notions, and it is always easy to recognize a contradiction as such. In practically all cases it can be brought into the form $1 = 2$. As a simple example, let us consider the proposition $\mathfrak{p} \equiv (\sqrt{2}$ is rational). It demands the construction of integers a, b, such that $a^2 = 2b^2$. By a well-known argument we may suppose that a and b are relatively prime. On the other hand, a is even, so 4 divides a^2, hence 4 divides $2b^2$, and b is even; thus a and b have the common divisor 2. This contradicts the fact that a and b are relatively prime. The contradiction can be given the form: The GGD of a and b is at the same time 1 and 2.

Some mathematicians, and notably Griss, have raised objections against the use of contradiction in mathematical reasoning. I shall treat these objections in the next chapter; here I take the point of view that the notion of a contradiction is sufficiently clear and that the negation which is based on it can be used in mathematics.

The *implication* $\mathfrak{p} \to \mathfrak{q}$ can be asserted, if and only if we possess a construction $\mathfrak{r}$, which, joined to any construction proving $\mathfrak{p}$ (supposing that the latter be effected), would automatically effect a construction proving $\mathfrak{q}$. In other words, a proof of $\mathfrak{p}$, together with $\mathfrak{r}$, would form a proof of $\mathfrak{q}$.

Almost every proof in this book consists of such a construction as $\mathfrak{r}$ above. One of the first instances, and a very clear one, is the proof of 2.2.3, Th. 2.

A logical formula with proposition variables, say $\mathfrak{A}(p, q, \ldots)$, can be asserted, if and only if $\mathfrak{A}(\mathfrak{p}, \mathfrak{q}, \ldots)$ can be asserted for arbitrary propositions $\mathfrak{p}, \mathfrak{q}, \ldots$; that is, if we possess a method of construction which by specialization yields the construction demanded by $\mathfrak{A}(\mathfrak{p}, \mathfrak{q}, \ldots)$. For example consider

$$\mathfrak{A}(p, q) \equiv (p \wedge \cdot\ p \rightarrow q\ \cdot \rightarrow q).$$

$\mathfrak{A}(\mathfrak{p}, \mathfrak{q})$ demands a construction E, which from a proof C of $\mathfrak{p}$ and a proof D of $\mathfrak{p} \rightarrow \mathfrak{q}$ yields a proof of $\mathfrak{q}$. By the definition of implication, E consists simply in the juxtaposition of C and D. Thus $\mathfrak{A}(p, q)$ can be asserted.

In 2.2.2 I gave a criterion for mathematical propositions, namely that every mathematical proposition has the form "I have effected a construction with the following properties: .....". This form is preserved by the four logical connectives. It is convenient to understand the word "construction" in the wider sense, so that it can also denote a general method of construction, as was meant in the last paragraph but one. If I do this – and I shall do it –, every logical formula expresses a mathematical proposition.

7.1.2. *Theorems of the propositional calculus*

In the formulas I use points and brackets in the usual way, assuming the convention that $\rightarrow$ binds less strongly that $\wedge$ and $\vee$. Asserted formulas are marked with $\vdash$.

Though the main differences between classical and intuitionistic logic are in the properties of the negation, they do not coincide completely in their negationless formulas. $p \rightarrow q \cdot \vee \cdot q \rightarrow p$ is a valid formula in classical logic, but it cannot be asserted in intuitionistic logic, as is clear from the definitions.

In the theory of negation the principle of the excluded middle fails. $p \vee \neg\, p$ demands a general method to solve every problem, or more explicitly, a general method which for any proposition $\mathfrak{p}$ yields by specialization either a proof of $\mathfrak{p}$ or a proof of $\neg\,\mathfrak{p}$. As we do not possess such a method of construction, we have no right to assert the principle.

Another form of the principle is $\neg\neg\, p \rightarrow p$. We have met many examples of propositions for which this fails: the first was "ϱ is rational" in 2.2.2. However,

(1) $\vdash p \rightarrow \neg\neg\, p.$

It is clear that from $\mathfrak{p}$ it follows that it is impossible that $\mathfrak{p}$ is impossible. I leave it to you to describe completely the method of construction which is demanded by (1), according to the definitions of $\rightarrow$ and $\neg$.

Another important formula is

(2) $\quad \vdash p \rightarrow q \cdot \rightarrow \cdot \neg q \rightarrow \neg p.$

Of course, the inverse formula, $\neg q \rightarrow \neg p \cdot \rightarrow \cdot p \rightarrow q$, is not assertable. (Take $\mathfrak{q} \equiv a \# b$, $\mathfrak{p} \equiv a \neq b$, where a and b are real numbers.)

Applying (2) twice, we find

(3) $\quad \vdash p \rightarrow q \cdot \rightarrow \cdot \neg\neg p \rightarrow \neg\neg q.$

By substitution in (1) we find

(4) $\quad \vdash \neg p \rightarrow \neg\neg\neg p.$

If we substitute $\neg\neg p$ for q in (2), we find, using (1),

(5) $\quad \vdash \neg\neg\neg p \rightarrow \neg p.$

(4) and (5) show that we need never consider more than two consecutive negations.

From $\vdash p \rightarrow p \vee q$ follows, by (2), $\vdash \neg (p \vee q) \rightarrow \neg p$; in the same way we have $\vdash \neg (p \vee q) \rightarrow \neg q$, so

(6) $\quad \vdash \neg (p \vee q) \rightarrow \neg p \wedge \neg q.$

The inverse formula is easily seen to be also true:

(7) $\quad \vdash \neg p \wedge \neg q \rightarrow \neg (p \vee q).$

(6) and (7) form one of de Morgan's equivalences. The other one is only half true:

(8) $\quad \vdash \neg p \vee \neg q \rightarrow \neg (p \wedge q).$

$\neg (p \wedge q) \rightarrow \neg p \vee \neg q$ cannot be asserted, as the following example shows. Let $\mathfrak{p}$ be $a \neq 0$ and $\mathfrak{q}$ be $b \neq 0$, where a and b are real numbers; then $\neg \mathfrak{p}$ is $a = 0$ and $\neg \mathfrak{q}$ is $b = 0$. I proved in 2.2.5 that $ab \neq 0$ is equivalent to $\mathfrak{p} \wedge \mathfrak{q}$, so $ab = 0$ is equivalent to $\neg (\mathfrak{p} \wedge \mathfrak{q})$: but just before the cited place in 2.2.5 I gave an example of real numbers a and b for which $ab = 0$, but neither $a = 0$ nor $b = 0$ is known.

(9) $\quad \vdash \neg\neg (p \vee \neg p).$

For $\neg (\mathfrak{p} \vee \neg \mathfrak{p})$ would imply, by (6), $\neg \mathfrak{p} \wedge \neg\neg \mathfrak{p}$, which is a contradiction. (8) gives by means of (2) and (6)

$$\vdash \neg\neg (p \wedge q) \rightarrow \neg (\neg p \vee \neg q) \rightarrow \neg\neg p \wedge \neg\neg q.$$

(10) $\vdash \neg\neg (p \wedge q) \rightarrow \neg\neg p \wedge \neg\neg q.$

The inverse formula is also true:

(11) $\vdash \neg\neg p \wedge \neg\neg q \rightarrow \neg\neg (p \wedge q).$

For it is clear from the above interpretation of the logical connectives that $\vdash \neg (p \wedge q) \wedge p \rightarrow \neg q$; then also $\vdash \neg (p \wedge q) \wedge \neg\neg q \rightarrow \neg p$. So, if $\neg\neg \mathfrak{p} \wedge \neg\neg \mathfrak{q}$ is given, the hypothesis $\neg (\mathfrak{p} \wedge \mathfrak{q})$ would lead to $\neg \mathfrak{p}$, which is contradictory with the given $\neg\neg \mathfrak{p}$.
It is easy to see that

(12) $\vdash \neg\neg p \vee \neg\neg q \rightarrow \neg\neg (p \vee q),$

but the inverse implication does not hold because of the strong interpretation of $\vee$.

7.1.3. *A formal system*

The intuitionistic propositional calculus has been developed [A. Heyting 1930] as a formal system with $\wedge$, $\vee$, $\rightarrow$, $\neg$ as undefined constants, and on the basis of the following axioms

I. $\vdash p \rightarrow (p \wedge p).$

II. $\vdash (p \wedge q) \rightarrow (q \wedge p).$

III. $\vdash (p \rightarrow q) \rightarrow ((p \wedge r) \rightarrow (q \wedge r)).$

IV. $\vdash ((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r).$

V. $\vdash q \rightarrow (p \rightarrow q).$

VI. $\vdash (p \wedge (p \rightarrow q)) \rightarrow q.$

VII. $\vdash p \rightarrow (p \vee q).$

VIII. $\vdash (p \vee q) \rightarrow (q \vee p).$

IX. $\vdash ((p \rightarrow r) \wedge (q \rightarrow r)) \rightarrow ((p \vee q) \rightarrow r).$

X. $\vdash \neg p \rightarrow (p \rightarrow q).$

XI. $\vdash ((p \rightarrow q) \wedge (p \rightarrow \neg q)) \rightarrow \neg p.$

The rules of deduction are the usual ones from the classical propositional calculus.

Axiom X may not seem intuitively clear. As a matter of fact, it adds to the precision of the definition of implication. You remember that $\mathfrak{p} \rightarrow \mathfrak{q}$ can be asserted if and only if we possess a construction which, joined to the construction $\mathfrak{p}$, would prove $\mathfrak{q}$. Now suppose that $\vdash \neg\, \mathfrak{p}$, that is, we have deduced a contradiction from the supposition that $\mathfrak{p}$ were carried out. Then, in a sense, this can be considered as a construction, which, joined to a proof of $\mathfrak{p}$ (which cannot exist) leads to a proof of $\mathfrak{q}$. I shall interpret the implication in this wider sense.

A system of intuitionistic logic in which $\rightarrow$ is interpreted in the narrower sense and in which, accordingly, X is rejected as an axiom, has been developed by Johansson in his "minimal calculus" [I. Johansson 1936].

It must be remembered that no formal system can be proved to represent adequately an intuitionistic theory. There always remains a residue of ambiguity in the interpretation of the signs, and it can never be proved with mathematical rigour that the system of axioms really embraces every valid method of proof.

7.2. The first order predicate calculus

7.2.1. *Interpretation of the quantifiers*

Let $\mathfrak{p}(x)$ be a predicate of one variable x, this variable ranging over a given mathematical species Q. Then $\vdash (\forall x)\mathfrak{p}(x)$ means that $\mathfrak{p}(x)$ is true for every x in Q; in other words, we possess a general method of construction which, if any element a of Q is chosen, yields by specialization the construction $\mathfrak{p}(a)$. In the case that Q is a spread-species, we must be able to effect the construction $\mathfrak{p}(x)$ for every ips x in Q; in the proof of the fan-theorem we saw that this is a very strong interpretation of the generalizing quantifier. The existential quantifier will also be interpreted in a strong way. $(\exists x)\mathfrak{p}(x)$ will be true if and only if an element a of Q for which $\mathfrak{p}(a)$ is true has actually been constructed.

The introduction of predicates with more than one argument presents no difficulty. A formula of the first order predicate calculus, which contains propositional and predicate variables, can be asserted if it is true for every substitution of propositions and predicates for these variables. A simple formalization of the intuitionistic predicate calculus is obtained by adjoining to the