Proofs as Constructions

Evidence in Euclidean Plane Geometry

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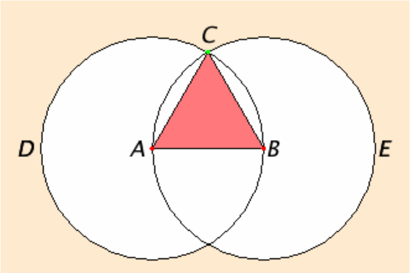
September 11, 2018

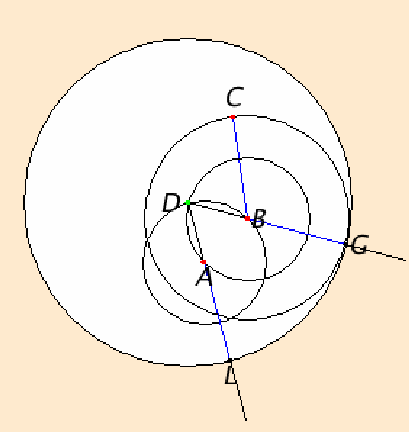
Euclid's Propositions

Proposition 1 : To construct an equilateral triangle on a finite straight line.



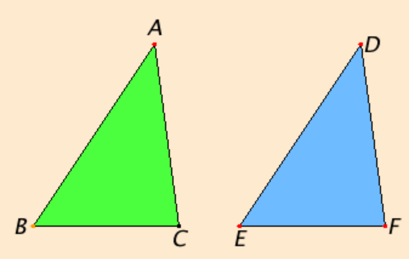
Euclid's Propositions

Proposition 2 : To place a straight line equal to a given straight line with one end at a given point.



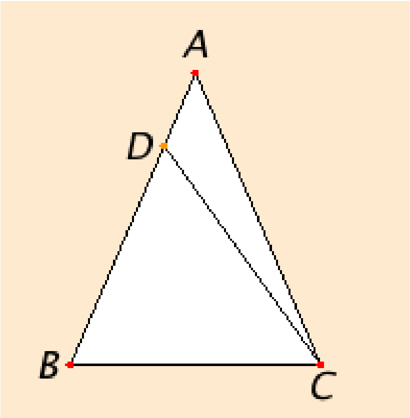
Euclid's Propositions

Proposition 4 : If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.



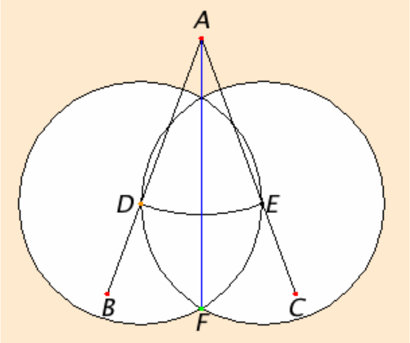
Euclid's Propositions

Proposition 6 : If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.



Euclid's Propositions

Proposition 9: To bisect a given rectilinear angle.



Euclid's Propositions

How can we explain the *meaning* of these propositions?

- Proposition 1 : To construct an equilateral triangle on a finite straight line.
- Proposition 2 : To place a straight line equal to a given straight line with one end at a given point.
- Proposition 4 : If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.
- Proposition 6 : If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.
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Euclid's Propositions

Problems and Theorems

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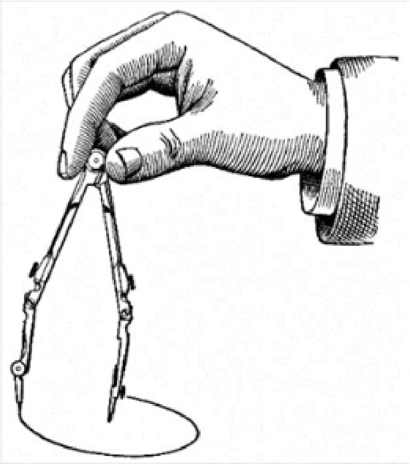
- The evidence for a *theorem* is the demonstration that certain relations hold for a given geometric object.

Euclid's Postulates



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3. To describe a circle with any center and distance
   · specifies a function with a point and finite straight line (distance) as input and a circle as output

Evidence Semantics

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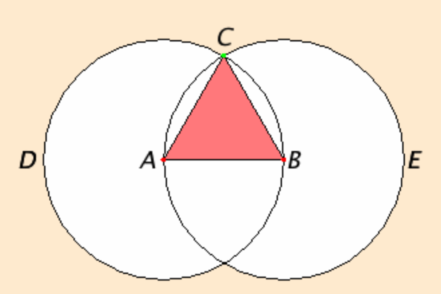
Proposition	Evidence Type	Evidence Term
$A \Rightarrow B$	function space: $[A] \to [B]$	$\lambda a.b$
$A \wedge B$	Cartesian product: $[A] \times [B]$	(a, b)
$A \vee B$	disjoint union: $[A] + [B]$	$inl(a), inr(b)$
$\forall x : A.B(x)$	dependent fun space: $x : [A] \to [B(x)]$	$\lambda a.b$
$\exists x : A.B(x)$	dependent product: $x : [A] \times [B(x)]$	(a, b)
$\bot$	empty set : $\{\}$	0

What is a logical form for Euclid's propositions?

Proposition 1 : To construct an equilateral triangle on a finite straight line.



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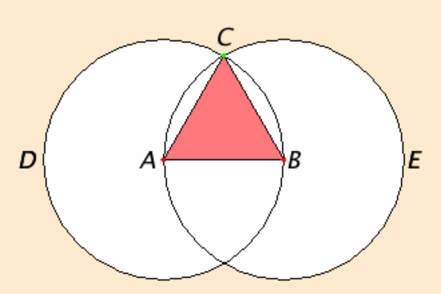

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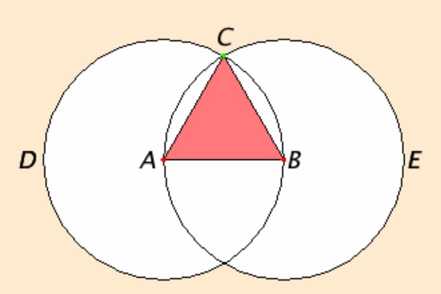

Think of it in terms of evidence:

the *evidence* for this proposition is a *function* taking a straight line into a *pair* consisting of a geometric object and a proof that the geometric object is an equilateral triangle.

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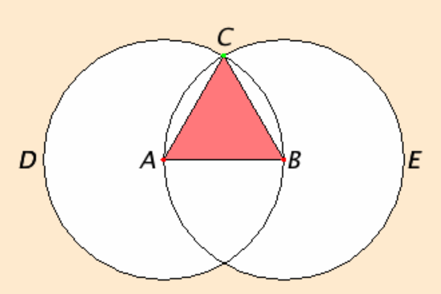

Think of it in terms of evidence:

$$l : line \to (t : \Delta(l) \times R(line, \Delta(l)))$$

General Forms for Euclid's Propositions

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  $\forall x : A \; \exists y : B(x).C(x, y)$
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  $\forall x : A.B(x)$

Formalizing the Straightedge and Compass

Primitive Objects and Relations

Definition

A *Euclidean Plane* structure has a primitive type Point together with the following relations for any $a, b, c, d \in$ Point.

Congruence, written $ab \cong cd$, says that segments ab and cd have the same length.

Betweenness, written a_b_c, says that the point b lies between a and c. This relation is not *strict*, so b could be equivalent to either a or c.

Apartness is a binary relation, signified by $\#$, on points. If $a \# b$ we say that a is *separated* from b.

Leftness is a ternary relation on points, written a left of bc, and says that the point a is to the *left* of the line bc (by bc we mean the *directed* line from b to c).

The Postulates

Straightedge-Straightedge "SS"

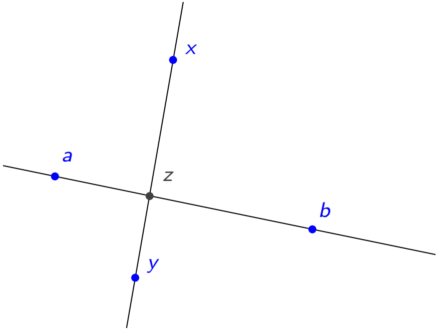


Figure: The Straightedge-Straightedge constructor for $z = SS(a, b, x, y)$.

The Postulates

Straightedge Compass "SC"

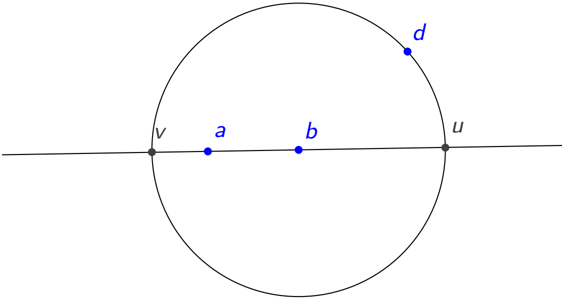


Figure: The Straightedge Compass constructor: $u = \mathrm{SCO}(a, b, c, d)$ and $v = \mathrm{SCS}(a, b, c, d)$

The Postulates

Compass-Compass "CC"

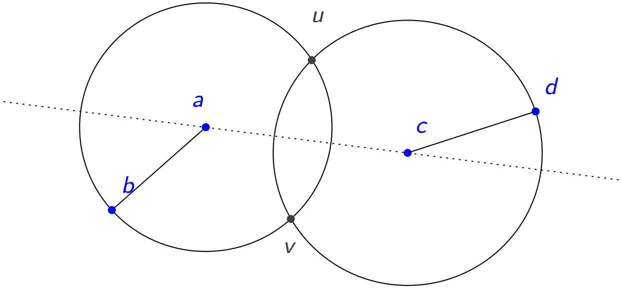


Figure: The Compass-Compass constructor for the points u and v such that u left of ac and v right of ac:
$u = \mathrm{CCL}(a, b, c, d)$ *and* $v = \mathrm{CCR}(a, b, c, d)$.

The Postulates

The *Magnifying glass*

$M(a, b, c)$ is the function that, by *magnification*, decides whether $c\#a$ or $c\#b$. Thus $M(a, b, c) \in c\#a + c\#b$.

The Postulates

Orientation

For any $a, b, c \in \text{Point}$, the function $\text{LeftOrRight}(a, b, c)$ takes evidence for $a \# bc$ into evidence for *orientation*:
$\text{LeftOrRight}(a, b, c) \in (a \text{ left of } bc + a \text{ right of } bc)$.

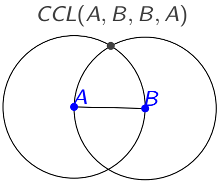
The Postulates

Non-triviality

guarantees that there exist two separated points, O and X such that $O \# X$.

Euclid's Proposition 1

To construct an equilateral triangle on a given finite straight line.



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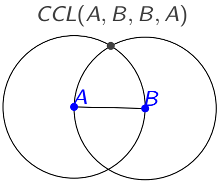

$$\forall A\!:\!\mathrm{Point}.\ \forall B\!:\!\{\mathrm{Point}|B\#A\}.\ \exists C\!:\!\{\mathrm{Point}|\ \mathrm{Cong3}(A,B,C)\wedge C \text{ left of } AB\}$$

where $\mathrm{Cong3}(A,B,C) := AB \cong BC \ \wedge\ BC \cong CA \ \wedge\ CA \cong AB$

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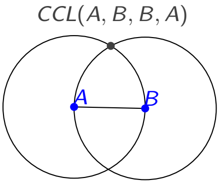

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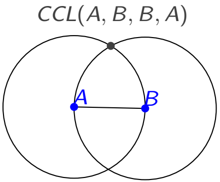

$$\forall A\colon \text{Point}.\ \forall B\colon \{\text{Point} \mid B \# A\}.\ \exists C\colon \{\text{Point} \mid \text{Cong3}(A, B, C) \land C \text{ left of } AB\}$$

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Our Nuprl extract reflects the simplicity of the construction:

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So we can define

$$\triangle(A, B) = CCL(A, B, B, A)$$

as the program for Euclid's Proposition 1.

Euclid's Proposition 2 (Lemma)

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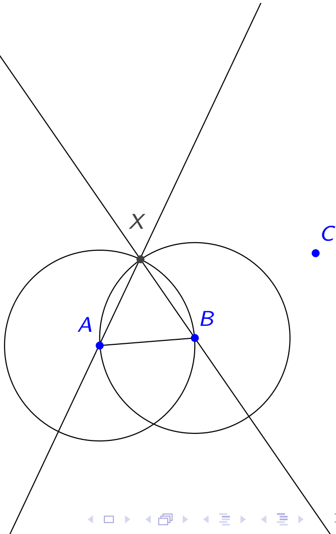

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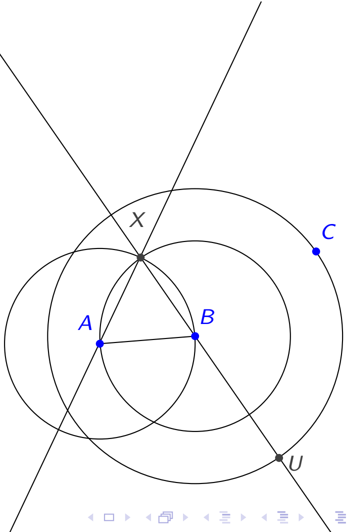
let $U = \text{SCO}(X, B, B, C)$



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L2(A,B,C) =

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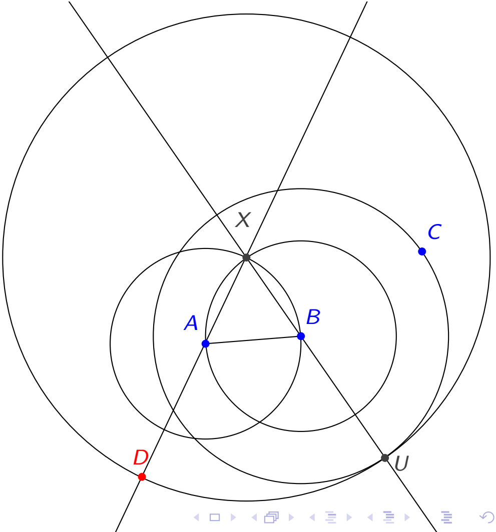

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Proof.

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case 1 $A \# B$: Apply L2(A, B, C) to get L.

case 2 $A' \# B$: Apply L2(A', B, C) to get L' then apply L2(A, A', L') to get L.

□

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$$\forall A, B, C : \text{Point}.\ \exists D : \{\text{Point} | AD \cong BC\}$$

$\text{Prop2(A,B,C)} = \text{if } M(O, X, A)$

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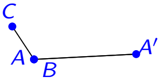

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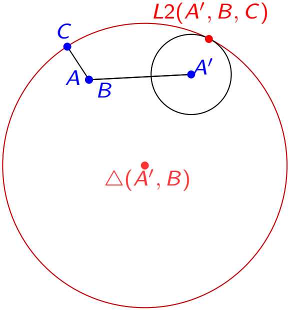

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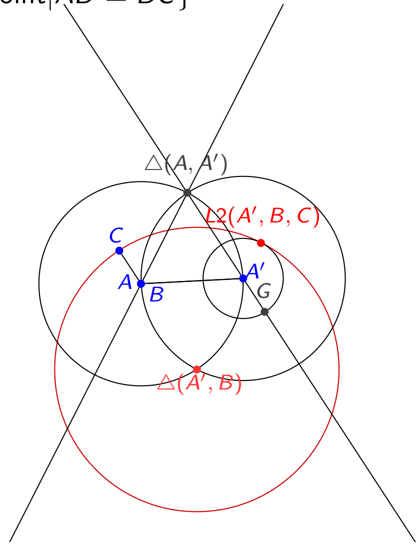

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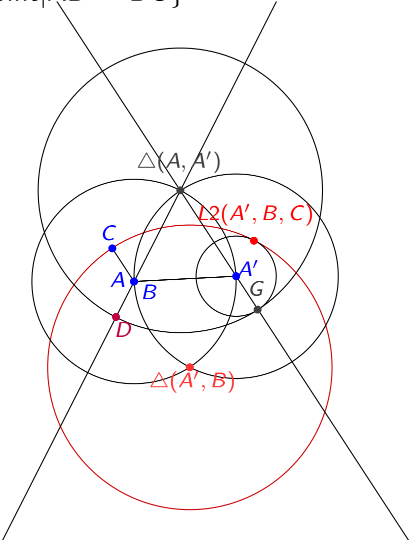

Closing Statements

- Interpreting the meaning of logical formulas by describing the evidence that can be given for them gives a faithful interpretation of Euclid's methodology of geometric constructions.
- This interpretation gives novel insights into Euclid's geometry (Prop 2).
- Both geometric construction problems and programming problems require the construction of a (computable) function that transforms an object of the input type into an object of the output type that satisfies some property.
- Thus, the semantics introduced here are generally suitable for expressing constructions in mathematics and computer science.