# CS/Math-4860fa-Lecture 2

August 23, 2018

**Abstract**

In this lecture there is a summary of Lecture 1, and we look briefly at the history of logic, from Aristotle's *Organon* (approx 330 BCE) and Euclid's, *Elements* (306 BCE) to the writings of Leibniz (1660's) [14], Frege's *Begriffsschrift* [9], Whitehead and Russell's *Principia Mathematica*, and Brouwer's *intuitionism* [11]. Later in the course we will study the implementation of many of these ideas in modern proof assistants.

## 1 Lecture 1 Summary

In addition to providing the Lecture 1 course summary, the first lecture stressed the increasing importance of logic in computer science because of software implementations of logic in modern *proof assistants* such as Agda, Coq, Nuprl and others to be mentioned during the course. This lecture gave a high level account of a very important new formal result in mathematics achieved by Dr. Mark Bickford using the Nuprl system [2]. Dr. Bickford was able to formally confirm the validity of a new mathematical principle proposed by the Field's Medalist Vladimir Voevodsky – the youngest Field's Medalist so far. This was an eighteen month effort that could not be formalized in Coq as Voevodsky [22] recognized when he turned to Cornell for help. We received NSF funding for this research as acknowledged in the article. This major result allows Nuprl to define Cubical Type Theory [4] inside Nuprl's Constructive Type Theory (CTT [1]). Later in the course we will discuss the impact of this important result and briefly discuss Cubical Type Theory, a current advanced research topic in logic and computer science – to which students who took this course two years ago are now contributing as CS graduate students at Carnegie Mellon University (CMU).

The lecture also discussed the fact that several modern proof assistants implement *type theory* rather than *set theory*. Most mathematicians prefer set theory as a foundation. One of the results we might discuss in the course is how type theory can define and implement set theory. Type theory arose in part from the fundamental work of Russell and Whitehead in defining a classical type theory in their massive work, *Principia Mathematica* [23]. The story of this great adventure is dramatized in the *Logicomix: An Epic Search for Truth* [6]. This is recommended reading for those who want to understand the cultural and historical background of some of the basic results we will cover in the course.

This first lecture pointed out that we learn certain basic elements of logic as children. It is needed to act in the world and is in some sense implicit in natural language expressions.

Finally, we mentioned the textbook, *First-Order Logic* [17]. This is one of the most compact and clear logic books ever written, by a prolific author who was also a magician. Everything he wrote is worth reading. Some other examples are *5000 B.C. and Other Philosophical Fantasies* [19] and *What is the name of this book??* [18] and *Gödel's Incompleteness Theorems* [20], and *Diagonalization and self-reference* [21].

The lecture mentioned that one of themes of the course will be an examination of the fact that proof assistants are becoming increasingly effective. In some fundamental sense "we are making them smarter." This theme gives certain elements of this course an artificial intelligence (AI) flavour.

# 2 Aristotle

Historians cite Aristotle's *Organon* as the first logic book, probably written at the Lyceum in Athens in the time of Ptolemy I circa 330 BCE. Aristotle presents patterns of inference and points out that a demonstration (proof) ends when we reach *nous*, which means intuition or insight. An example of such a demonstration is that from the premise *All people are mortal*, and the premise, *Socrates is a person*, we judge that *Socrates is mortal.*

In modern logical notation we would write "all people are mortal" as $\forall x : Person.Mortal(x)$, and $soc\ \epsilon Person$, we conclude $Mortal(soc)$.

The general form of this argument is $\forall x : D.P(x)$ and $d\epsilon D$, we know $P(d)$. We call $D$ the *domain of discourse.* The quantifier $\forall x$ means "for all x."

# 3 Euclid

Euclid's *Elements* [8, 7, 10] is a classic work of mathematics in all senses of the word. It remains widely read, new editions have been created as late the 2007 citation, and new concepts are discovered to provide alternative proofs of the ancient propositions. For this course, one of the most interesting features of the *Elements* is that the proofs are *constructive*, and therefore they teach elements of *constructive mathematics.* One of the key features of such mathematics is that to say that an object exists is to show how to construct it, e.g. an equilateral triangle or a right angle or parallel lines.

We will use Euclid to illustrate the idea of constructive reasoning and constructive mathematics. From that simple basis we will be able to explore with confidence the idea of constructive reasoning in other areas of mathematics and especially in computer science.

Geometry is the subject that two very distinguished mathematicians used to develop the axiomatic method. Hilbert provided a modern axiomatic account in his classic book *Foundations of Geometry* [12]. Tarksi and his colleagues W. Schwabhäuser, and Wanda Szmielew also axiomatized geometry, *Metamathematische Methoden in der Geometrie*, written in German [16]. This book has been formalized using the Coq proof assistant. Both of these relatively modern accounts use what we now call *classical mathematics.* We will use examples from geometry, number theory, and analysis to teach the notion of constructive proofs. Geometry provides an especially good subject for teaching modern constructivity.

## 4 Leibniz

Gottfried Wilhelm Leibniz (1676 - 1716) was co-inventor of the calculus with Newton. We still use his notation for operations in calculus. He also created notations for logical reasoning and imagined reductions of reasoning to *calculating with logical notation*. He put it this way in a famous quote from his 1666 book *De Arte Combinatoria*: “The only way to rectify our reasonings is to make them as tangible as those of the Mathematicians, so that we can find our error at a glance, and when there are disputes among persons, we can simply say: Let us calculate [calculemus], without further ado, to see who is right.”

One of my favorite sentences about the value of symbolic logic is from Leibniz. I have quoted it in articles. Here it is again: “We can judge immediately (even mechanically) whether propositions presented to us are proved, and that which others could hardly do with the greatest mental labor and good fortune, we can produce with the guidance of symbols alone ... as a result of this, we shall be able to show within a century what many thousands of years would hardly have granted to mortals otherwise.” [14]

Leibniz also is a major node in the genealogical tree of mathematicians and logicians. Many logicians since the 1660s can trace their academic ancestry back to Leibniz, including me.

## 5 Frege

Many historians believe that the most significant achievement in logic after Aristotle is Frege’s publication in 1879 of his *Begriffsschrift* (concept-script, concept-writing, idea-writing, ideography) [9, 13]. It is a booklet of 88 pages that revolutionized logic.

In the *Begriffsschrift, A Formula Language, modeled Upon that for Arithmetic for Pure Thought* Frege invented what we now call *first-order logic* (FOL), a major topic of this course. He started by making a case for advancing Leibniz’s conception of logic and arguing that the Aristotelian conception was too narrowly tied to natural language. He said: “These derivations (in notation) from what is traditional find their justification in the fact that logic has hitherto always followed ordinary language and grammar too closely. In particular, I believe that the replacement of the concepts subject and predicate by argument and function, respectively, will stand the test of time.”

Frege was attempting to understand the notion of a sequence very precisely. The gradual arithmetization of analysis and the calculus (see below), had reduced the issue to understanding the natural numbers. Frege says his:

“Leibniz, too, recognized and perhaps overrated the advantages of an adequate system of notation. His idea of a universal characteristic, of a calculus philosophicus or ratiocinator, was go gigantic that the attempt to realize it could not go beyond the bare preliminaries. The enthusiasm that seized its originator when he contemplated the immense increase in the intellectual power of mankind that a system of notation directly appropriate to objects themselves would bring about led him to underestimate the difficulties that stand in the way of such an enterprise. But, even if this worthy goal cannot be reached in one leap, we need not despair of a slow, step-by-step approximation. When a problem appears to be unsolvable in its full generality, one should temporarily restrict it; perhaps in can then be conquered by a gradual advance."

# 6 Russell

In 1902, Bertrand Russell discovered a paradox along similar lines as Cantor's, but much simpler to explain. He considered the set of all sets that do not contain themselves, call it R and define it as $\{x|\neg(x \in x)\}$. Now Russell asks whether $R \in R$. We can see immediately that if $R \in R$, then by definition of $R$, we know $\neg(R \in R)$. If $\neg(R \in R)$, then by definition of $R$, $R \in R$.. Since according to classical logic either $R$ belongs to $R$ or it does not, and since each possibility leads to a contradiction, we have a contradiction in classical mathematics if we assume there is such a set as R. This is known as *Russell's Paradox.*

In trying to sort out the reasons behind this paradox, Russell was led to formulate his famous Theory of Types in the 1908 book *Mathematical Logic Based on a Theory of Types* [15]. Then in the period from 1910 to 1925 Whitehead and Russell wrote and published a three volume book based on this type theory and designed to be a secure logical foundation for all of mathematics entitled *Principia Mathematica* (PM) [23]. We will examine aspects of this type theory during the course, and we will use type theory from the beginning in our informal mathematics because the notion of type is now central in computer science, owing largely to early research in Britain, France, Sweden and the US as well as to the increasing role of type systems in the design of programming languages. The book *Logicomix: An Epic Search for Truth* [6] is a comic book style account of the creation of *Principia.* A more mundane article on this topic is entitled *The Triumph of Types* [5].

# 7 Krönecker

Krönecker was a distinguished German mathematician known for his work in algebraic and analytic number theory. He is also known for being opposed to using the concept of a completed infinite set in mathematical reasoning as Cantor and others were starting to do. He agreed with Gauss that the idea of an infinite set was just a manner of speaking about collections such as the natural numbers, 0;1;2;3; ... which can be continued without end because given any number n, we can construct a larger number by adding one to it. He was in particular opposed to Cantor's work on set theory, but he also disagreed with methods of proof that did not produce concrete answers. He strongly favored computational methods and explicit constructions.

Kröneckeris mentioned in Bell's *Men of Mathematics* as being "viciously opposed" to Cantor and others who proved results about infinite sets as if they were completed totalities, but this appears to be a considerable exaggeration. What is true is that in 1886 he made an after dinner speech in which he said "God made the integers, all else is the work of man" as a way of expressing his interest in constructions. Here is the German for what he said:

"Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk." We will look next at a mathematician who did openly attack non-computational methods and who was in contention for being the leading mathematician of his day.

# 8 Brouwer

In his 1907 doctoral dissertation, *On the Foundations of Mathematics* [3], Brouwer provided a meaning for mathematical statements based on mental constructions. These constructions are intuitively known to be effective for basic mathematical tasks. This philosophical stance is known as *intuitionism*, and Brouwer was interested in building mathematics according to this philosophy. Brouwer believed that the crisis in the foundations of analysis was due to mathematicians not understanding the full extent of constructive methods. In particular he believed that our mental constructions are the proper justification for logic and that they do not justify even full first-order logic.

One of the fundamental logical laws that is not justified according to Brouwer is the *law of excluded middle*, P or not P, for any proposition P. For Brouwer, to assert P is to know how to prove it, and he could imagine propositions which we could never prove or disprove. Brouwer believed that logic was the study of a particular subset of abstract constructions but that it played no special role in the foundations of mathematics, it was simply a form of mathematics.

Brouwer believed that our mathematical intuitions concern two basic aspects of mathematics, the discrete and the continuous. Our intuitions about discreteness and counting discrete objects give rise to number theory, and our intuitions about time give rise to the notion of continuity and real numbers. He believed that mathematicians had not recognized the rich constructions need for understanding real numbers, including for computing with real numbers. He was determined to study these constructions and thus settle the disputed issues in analysis and in the theory or real numbers. But first he decided he should establish himself as the best mathematician in the world.

He proceeded to develop point set topology and proved his famous and widely used *fixed point theorem*. He attracted several converts and fellow travelers, and when one of the most promising young mathematicians, Weyl, became a follower of Brouwer, another contender for "world's best mathematician", David Hilbert, entered the fray in a "frog and mouse war" with Brouwer. Hilbert is famous for saying that "we can know and must know" the truth or falsity of any mathematical statement. He is also famous for *Hilbert spaces* and many more fundamental concepts.

# References


[1] Stuart Allen, Mark Bickford, Robert Constable, Richard Eaton, Christoph Kreitz, Lori Lorigo, and Evan Moran. Innovations in computational type theory using Nuprl. *Journal of Applied Logic*, 4(4):428–469, 2006.

[2] Mark Bickford. Formalizing category theory and presheaf models of type theory in nuprl. Technical Report arXiv:1806.06114v1v1, CS.LO 15 June 2018, 2018.

[3] L.E.J. Brouwer. *Over de grondslagen der wiskunde (On the foundations of mathematics)*. PhD thesis, Amsterdam and Leipzig, 1907.

[4] Cyril Cohen, Thierry Coquand, Simon Huber, and Anders Mörtberg. Cubical Type Theory: a constructive interpretation of the univalence axiom, 2015. Preprint.

[5] Robert Constable. The triumph of types: Principia mathematica's impact on computer science. In *Principia Mathematica anniversary symposium*, 2010.

[6] A Doxiadis and C. Papadimitriou. *Logicomix: An Epic Search for Truth*. Ikaros Publications, Greece, 2008.

[7] Euclid. *The Elements*. Green Lion Press, Santa Fe, New Mexico.

[8] Euclid. *Elements*. Dover, approx 300 BCE. Translated by Sir Thomas L. Heath.

[9] Gottlob Frege. Begriffsschrift, a formula language, modeled upon that for arithmetic for pure thought. In J. van Heijenoort, editor, *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*, pages 1–82. Harvard University Press, Cambridge, MA, 1967.

[10] T. Heath. *The thirteen books of Euclid's Elements*. Dover, New York, 1956.

[11] A. Heyting, editor. *L. E. J. Brouwer Collected Works*, volume 1. North-Holland, Amsterdam, 1975.

[12] David Hilbert. *Foundations of Geometry*. Open Court Publishing, 1992.

[13] Anthony Kenny. *Frege*. Penguin Books, London, 1995.

[14] Gottfried Leibniz. *Logical Papers: A Selection*. Clarendon Press, Oxford, 1966.

[15] Bertrand Russell. Mathematical logic as based on a theory of types. *Am. J. Math.*, 30:222–62, 1908.

[16] W. Schwabhäuser, Wanda Szmielew, and Alfred Tarski. *Metamathematische Methoden in der Geometrie*. Springer Verlag, Berlin, 1983.

[17] R. M. Smullyan. *First–Order Logic*. Springer-Verlag, New York, 1968.

[18] R. M. Smullyan. *What is the name of this book??* Pelican, 1984.

[19] Raymond M. Smullyan. *5000 B.C. and Other Philosophical Fantasies*. St. Martin's Press, NY, 1983.

[20] Raymond M. Smullyan. *Gödel's Incompleteness Theorems*. Oxford University Press, New York, 1992.

[21] Raymond M. Smullyan. *Diagonalization and self-reference*. Number 27 in Oxford Logic Guides. Clarendon Press, Oxford, 1994.

[22] Valdimir Voevodsky. Notes on type systems. School of Math, IAS, Princeton, NJ, 2011.

[23] A.N. Whitehead and B. Russell. *Principia Mathematica*, volume 1, 2, 3. Cambridge University Press, 2nd edition, 1925–27.