CHAPTER 15

INTUITIONISM

131. Introduction

Intuitionism, though it was anticipated by Kant and, in recent times, by mathematicians such as L. Kronecker and H. Poincaré, was not developed in a systematic and consistent manner before the work of Brouwer and his school. It constitutes a tendency of its own within the vast domain of research into the foundations of mathematics and should be approached as such, if it is to be properly understood. This may be illustrated by quoting some of its most important and distinctive maxims.

(1) It is not possible to penetrate the foundations of mathematics without paying attention to the conditions under which the mental activity proper to mathematicians takes place.

(2) Research which does not give proper attention to this side of the problem is unable to reveal the essence of mathematical thinking; it can give information only as to its external appearance.

[It may be recalled that Frege, on the contrary, insisted that we have no chance of approaching the essence of mathematics as long as we indulge in psychological speculations about mathematical thinking.]

(3) Not only is research into the foundations of mathematics often inspired by false conceptions of mathematical entities and mathematical activity; but the same is also true for much mathematical research in the ordinary sense of the word.

Consequently, research into the foundations of mathematics cannot take existing mathematical theories for granted and restrict itself to the provision of a suitable foundation for these theories; it must submit these theories to a penetrating criticism.

(4) Mathematics should be developed independently of any preconceived ideas of the nature of the mathematical entities or of mathematical activity. Mathematical theories which depend essentially upon such preconceived ideas cannot sustain intuitionistic criticism.

(5) Mathematics is independent of logic, logic itself being dependent upon mathematical thinking. The currently accepted principles of logic do not deserve the unrestrained confidence usually given to them.

In the following sections we shall explain the far-reaching implications of these methodological principles as set forth by Brouwer and his followers.

132. Intuitionistic Criticism of Current Mathematics and of the Currently accepted Methods for Research into its Foundations

Already in his thesis for the doctor's degree (1907) Brouwer submitted to a penetrating criticism (1) the axiomatisation of mathematics, (2) Cantor's theory of sets, (3) symbolic logic as developed by Peano and Russell, and (4) Hilbert's ideas concerning the foundations of mathematics.

It should be borne in mind that, at the date of the publication of Brouwer's thesis, these doctrines had not yet obtained their present form. On the other hand, Brouwer himself did not at that time realise all the implications of his own views. Nevertheless, he shows a deep insight into many essential features of the doctrines which he rejects; moreover, the ideas which he expounded in his thesis are deserving of special attention as they form the historical background to the later development of intuitionistic mathematics.

(1) Axiomaticians are reproached with establishing merely verbal edifices without paying due attention to the construction of a corresponding system of mathematical entities. Moreover, they are charged with inconsistency in that they falsely accept the consistency of an axiom system as a guarantee of the existence of a system of mathematical entities fulfilling the axioms, while at the same time they appeal to the existence of intuitively constructed systems of mathematical entities in their proofs of consistency.

(2) Set theory as expounded by Cantor and Zermelo is practically entirely rejected by Brouwer. He accepts neither the theorem of Bernstein–Schröder, nor the axiom of choice and the well-ordering theorem based on it. Later he developed a theory of sets on intuitionistic lines; this theory will be explained later in this chapter.

(3) With regard to symbolic logic, Brouwer's opinion is that it can teach us nothing about mathematics, as it is condemned to

remain for ever separate from mathematics. It is only a faithful, mechanical, stenographic imitation of mathematical language; this language itself. as a matter of fact, does not belong to mathematics proper; it is nothing more than an imperfect tool, used by mathematicians to communicate their results and to render them more easily retained. Logic, traditional as well as symbolic, is an empirical science, belonging rather to ethnography than to psychology.

(4) In his criticism of Hilbert's ideas, Brouwer gives a striking description of the successive stages in the formalisation of mathematics. He enumerates: (i) the construction of intuitive systems of mathematical entities; (ij) the verbal parallel of mathematical thinking, that is, mathematical language; (iij) the mathematical analysis of this language; this activity leads to the discovery of verbal edifices established in accordance with the principles of logic; (iv) the step of abstracting from the meaning of the elements which constitute these verbal edifices; the abstract systems thus obtained are considered to be mathematical systems of the second order; they are identical with the formal systems studied by symbolic logic; (v) the introduction of the language of symbolic logic which accompanies logical constructions; this stage is found in the works of Peano and Russell; (vi) the mathematical analysis of the language of logicians; this stage, initiated by Hilbert, had been neglected by Peano and Russell; (vij) the step of abstracting ... etc. – According to Brouwer, mathematics is only to be found in the first stage of the process; the second stage in unavoidable from a practical point of view; the later stages are of a derivative character.

In this analysis of the process of formalisation we find a strikingly clear insight into the necessity of a separation between mathematics and metamathematics. This insight was gained by Hilbert gradually in the period between 1900 and 1923.

133. Existence and Consistency

Brouwer's remarks concerning the subreptive character of the transition from the formal consistency of a set of axioms to the existence of a corresponding mathematical system admit a striking illustration from the discovery of axiom sets which can be shown to be formally consistent and which nevertheless cannot have a standard model. We have already discussed various examples of such a situation, but it is worth while briefly to consider still another case.

Let us take the full Dedekind–Peano set as described in Section 51 and let us add the axiom schema (Q0) as introduced in Section 97. It is easy to show by essentially the same argument as has been used in Section 97 that the resulting axiom set B is still formally consistent.

This proof of formal consistency is certainly conclusive from an intuitionistic point of view. And, from an intuitionistic point of view, it is also clear that the axiom set B cannot have a model $[N, f, e, A]$.

On the other hand, our example shows that, contrary to Brouwer's opinion, in some cases a proof of formal consistency can be given which does not involve the construction of a model for the axiom set under consideration. Indeed, our argument can be summed up as follows: (i) every finite subset B^0 of B has a model $[N, f, e, A]$; (ij) it follows that every finite subset B^0 of B is formally consistent; (iij) hence the set B itself must be formally consistent.

From a semantic point of view, our example proves that the predicate logic of higher order is incomplete: from the fact that a set of formulas from higher order logic has no model it does not follow that this set is formally inconsistent. – Leon Henkin (1950) has shown that, in a weaker sense, higher order logic can be proved to be complete (*cf.* Section 184). The notion of a model has to be extended; models in the ordinary sense are called *standard models*. It can then be proved that a set of formulas of higher order logic has a model, if and only if it is formally consistent. A similar result is implicitly contained in Mostowski's paper on absolute properties (1947). Its implications which respect to Brouwer's observations were pointed out in my *Fondements logiques des mathématiques* (1950).

134. Logic and Mathematics

On the basis of his schematisation of the constitution of formalised mathematics, Brouwer states that logic introduces itself only at the stages: (ij) creation of a mathematical language, and (iij) mathematical analysis of this language. According to Brouwer, however, these stages arise only out of practical needs; they do not represent any essential features of mathematical thinking. It follows that intuitive mathematics – stage (i) – is completely independent of logic. On the other hand, mathematical language, and hence logic also, is wholly dependent on the needs of intuitive mathematical thinking, to which, therefore, it must adapt itself as closely as possible. For this reason Brouwer fiercely contests what he calls "the illusion of the liberty of logic". So his conceptions are the exact opposite of the opinion of Carnap, who, in his "tolerance principle", gave a classical statement of the creed of the adherents to the unrestricted liberty of logic.

It will be clear from the preceding section that in modern formalisations of mathematics there is indeed a strong tendency to adapt logic as closely as possible to the needs of mathematics. The introduction of a new rule of inference to eliminate, as far as possible, the emergence of non-standard models for deductive theories may be interpreted as a symptom of this tendency.

Of course, there is an opposing tendency as well, which manifests itself in the construction of logical systems which, in one sense or another, diverge from "normal" logic.

However, in most cases, these systems are not meant actually to supersede "normal" logic; their construction is intended rather to provide a better insight into the foundations of logic. In some cases, it even serves the purpose of adapting logic better to the needs of mathematical reasoning.

So Brouwer's rejection of the liberty of logic does not, in itself, conflict as strongly with the general trend in the study of formalised mathematics as it might appear to do. There are, however, more radical implications of his conceptions which give Brouwer's intuitionism an isolated position in modern foundational research in mathematics.

135. The Principle of the Excluded Third

One of the most spectacular features in Brouwer's intuitionism is, of course, his rejection of the unrestricted application of the principle of the excluded third in mathematical reasoning. In order to give a reliable account of the development of intuitionistic mathematics, it will be necessary to give a full exposition of the reasons for this radical step.

In 1900, Hilbert stated the principle of the essential solvability of every mathematical problem. According to him, this principle embodied a common conviction of all mathematicians; in this sense the principle was again stated in 1906, by the philosopher Leonard Nelson.

Later, however, Hilbert came to realise that, with regard to formalised mathematics, the solvability principle, far from being the expression of an obvious insight, constituted a serious and highly interesting problem, which he called the *decision problem* ("Entscheidungsproblem"). and to which much research of fundamental importance has been devoted in recent years (*cf.* Section 212). We have seen that recent research has brought to light the existence of mathematical problems unsolvable in an absolute sense.

According to Brouwer, the application of the principle of the excluded third implies a tacit and, in general, unjustified appeal to the solvability principle. In order to make his views completely clear, it will be helpful to analyse some typical cases of the application of the principle of the excluded third in mathematical reasoning on the basis of a few preliminary remarks.

(1) In general, disjunction may be characterised by the following rules of inference:

$$\frac{P}{P \vee Q} \ \text{(a)} \qquad \frac{Q}{P \vee Q} \ \text{(b)} \qquad \frac{P \to R,\ Q \to R}{(P \vee Q) \to R} \ \text{(c)}$$

However, if only these rules were given, disjunction as a logical operation would be completely sterile. As a matter of fact, in order to derive a conclusion R by applying the rules (a)–(c), we should need, as premisses, one of the expressions P and Q, as well as both expressions $P \to R$ and $Q \to R$. However, by means of the *modus ponens*, R can be derived either from P and $P \to R$, or from Q and $Q \to R$ alone. The fertility of disjunction as a logical operation derives from the fact that, in addition to the rules (a)–(c), we also, in many cases, admit, without previous proof (that is, as axioms), expressions having the form of a disjunction. The most important example is, of course, the acceptance of the principle of the excluded third:

$$p \vee \overline{p} \quad \text{(d)},$$

which enables us to apply the secondary rule of inference:

$$\frac{P \to R,\ \overline{P} \to R}{R} \quad \text{(e)}$$

The rule (e) is called a secondary rule because it can be reduced to the rules (a)–(c) in combination with the principle (d).

(2) Now let us consider a simple case of the application of this rule of inference. Suppose that we want to prove:
7 divides

$$N^2 = (1 + 2^2 + 3^3 + \ldots + 99^{99})^2$$

if and only if 7 divides

$$N = 1 + 2^2 + 3^3 + \ldots + 99^{99}.$$

The obvious way of proceeding is as follows:

(i) Suppose that 7 divides N. Then we have, for some k, $N = 7 \cdot k$, $N^2 = 49 \cdot k^2$; so 7 divides N^2.

(ii) Suppose that 7 does not divide N. Then we have, for some k and q: $N = 7 \cdot k + q$, and $q = 1$, or $q = 2$, or $q = 3$, or $q = 4$, or $q = 5$, or $q = 6$. So we need only verify that 7 does not divide any of the numbers 1, 4, 9, 16, 25, 36.

It will be clear that the rules (a)–(c) do not constitute a sufficient basis for this argument. If we were to appeal only to these rules, it would be necessary to derive explicitly one of the suppositions (i) and (ii), that is, to solve the problem whether 7 divides N or not. This step can, however, be avoided by means of an appeal to principle (d). Thus we may, with regard to our example, distinguish two divergent attitudes.

(I) Rules (a)–(c) are adopted, but principle (d) is rejected. Then our argument must be completed by means of an explicit solution of the problem whether 7 divides N or not.

(II) In addition to rules (a)–(c), principle (d) is admitted. Then we need not go into the above-mentioned problem.

So the acceptance of principle (d) appears to imply that we take for granted the existence of a solution to this problem; in other words, the acceptance of principle (d) as a reliable starting-point for mathematical reasoning implies the acceptance of the principle of the essential solvability of every mathematical problem.

It will be clear that, besides the extreme attitudes described under (I) and (II), an intermediate attitude remains possible, namely

(III) Rules (a)–(c) are admitted, principle (d) is rejected, but, instead of an explicit solution of the problem whether 7 divides N or not, only sufficient evidence of the solvability of the problem is demanded.

In the present example, this attitude enables us to give a simple proof. Indeed, it is possible to carry out the calculation of the number N and then, by applying the division algorithm, to decide whether 7 divides N or not. It is certain in advance that we shall, in this manner, obtain an explicit proof either for supposition (i) or for supposition (ii).

(3) The attitude described under (I) fails to provide the basis for a mathematical proof when we wish to prove theorems of a more general nature, for instance:

Let p be any prime number, and N any natural number; then p will divide N^2, if and only if p divides N.

In this case, we cannot prove, in advance, either of the suppositions:

(i) p divides N;

(ii) p does not divide N.

Consequently, we are bound to adopt either attitude (II) or attitude (III). It will be clear that attitude (II) affords a basis for a proof. As to attitude (III), we get a proof by observing that, for any given values of p and N, we can decide whether (i) or (ii) is correct.

As a result of our discussion, we can state that, if attitude (III) is adopted, we are justified in applying principle (d) in those cases in which we are able to decide whether P or $\bar{P}$ holds good.

(4) In more complicated cases, neither attitude (I) nor attitude (III) affords a basis for a proof, even though attitude (II) does afford such a basis. For example, let us suppose that we want to prove:

For every real number a, there is a real number x, which satisfies the equation;

$$(1-a)\cdot x-\sin(1-a)=0.$$

The usual procedure would be as follows:

(i) Suppose that $a=1$, then we can take $x=1$.

(ii) Suppose that $a\neq 1$, then we can take $x=\sin(1-a)/(1-a)$.

Now if attitude (III) is adopted, we must complete this argument by pointing out a method which, for any given real number a, enables us to decide whether $a=1$ or $a\neq 1$.

However, we have no such method. Indeed, let a real number a be defined as follows:

If n is the smallest number k such that the kth digit d_k in the decimal representation of the real number $(\pi+e)^{\pi-e}$ is the first in a sequence of 100 digits, all equal to d_k, then $a=1-\frac{1}{10^n}$; if there is no such number n, then $a=1$.

By this definition a certain real number a is uniquely determined. For, in order to obtain the mth digit in the decimal representation of a, it is clearly sufficient to compute the first $m+100$ digits in the decimal representation of the real number $(\pi+e)^{\pi-e}$. On the other hand, there is at present no method which would enable us to find out whether $a=1$ or $a \neq 1$. So, if we persist in rejecting principle (d), our argument breaks down.

It should be noted that, nevertheless, attitude (III) still allows the provision of a proof, which is based on the obvious fact that the function, defined by:

$$y = \lim_{x \to a} \sin(1-x)/(1-x),$$

is continuous for $a=1$.

However, this argument, also, breaks down if we now consider the equation:

$$(1-a)\cdot x - \sin \sqrt{1-a} = 0.$$

(5) More involved situations of a similar nature arise when quantifiers are introduced. Let us consider, for instance, the logical principle:

$$(n)A(n) \vee (En)\overline{A(n)} \tag{f}$$

which we shall, for once, write as an infinite disjunction:

$$(n)A(n) \vee \overline{A(1)} \vee \overline{A(2)} \vee \ldots \vee \overline{A(k)} \vee \ldots.$$

If we still adopt attitude (III), then the application of this principle as a basis for mathematical proof will be justified only if, for a given expression $A(n)$, we have a method which enables us to single out a member of the infinite disjunction which can be shown to hold good.

In certain cases such a method is indeed available. For instance, let $A(n)$ stand for the phrase:

the natural number n does not divide the number N defined under (2), *if* $n \neq N$;

obviously, by checking, for every prime number $p \leqq \sqrt{N}$, whether or not N is divisible by p, we obtain a mathematical proof for one of the statements $(n)A(n)$, $\overline{A(1)}$, $\overline{A(2)}$, ..., $\overline{A(k)}$, ...

(6) However, let us now take for $A(n)$ the phrase:

the nth digit d_n in the decimal representation of the real number $(\pi+e)^{\pi-e}$ is not the first of a sequence of 100 *digits, all equal to d_n*;

then, obviously, the argument given in connection with the preceding example can no longer be maintained.

Example. Discuss the second equation under (4), adopting attitude (III).

136. Non-Constructive Existence Proofs

Intuitionistic criticism dislocates many arguments which at first glance may seem very simple and very safe. Let us consider, as an example, the so-called *least number principle*:

Suppose that a natural number n is given which has the property E; then there must also be a smallest number m having the property E.

Proof. We apply recursion on k to show that the principle holds good for every property E and for every $n \leqq k$.

(A) Suppose that a certain number $n \leqq 1$ has the property E. Then clearly 1 has the property E and is the smallest number having this property.

(B) Suppose that the principle holds good for every property E and for every $n \leqq k$; we have to show that it holds good for every property and for every $n \leqq k+1$. So assume that a certain number $n \leqq k+1$ has the property E; we consider three cases.

(i) $n=k+1$, and $k+1$ is the smallest number having the property E; in this case $m=k+1$;

(ij) $n=k+1$, but $k+1$ is not the smallest number having the property E; then there is a number $n<k+1$ which has the property E; this case reduces to case (iij);

(iij) $n<k+1$, hence $n \leqq k$; therefore, by the induction hypothesis there is a smallest number m having the property E.

From an intuitionistic point of view, this argument is not conclusive. In fact, from this point of view, the validity of the least number principle itself seems very doubtful, to say the least of it. Let E be the following property of a natural number n: there is, in the decimal representation of the real number $(\pi+e)^{\pi-e}$, a sequence of $100-n$ digits all equal to each other. The natural number $n=99$ obviously has the property E. However, there is at present no method at our disposal which enables us to answer the question whether or not 99 is the smallest number having the property E and, if not, which is the smallest number having this property.

Therefore, if interpreted in an intuitionistic sense, that is, in accordance with attitude (III), the least number principle cannot be considered to be satisfactorily established.

Example. Show that the least number principle holds good from an intuitionistic point of view, whenever E is a decidable property.

137. Technical Complications Deriving from the Acceptance of an Intuitionistic Conception of Mathematical Proof

Even the treatment of comparatively elementary problems meets with unexpected and considerable difficulties, when we adopt an intuitionistic point of view.

We have already devoted a few words to the discussion of the linear equation:

$$a\cdot x+b=0$$

in the real field.

In classical mathematics, one would say: there are three cases to be distinguished, namely:

(i) $a=0$, $b\neq 0$; in this case, no real value of x can ever satisfy the equation;

(ii) $a=0$, $b=0$; in this case, any real value of x will satisfy the equation;

(iii) $a\neq 0$; in this case, the real number $-\frac{b}{a}$ will be the only real value of x which satisfies the equation.

This manner of dealing with the problem is inadequate from an intuitionistic point of view. This will be clear when the values of the coefficients a and b are chosen as follows:

if n is the smallest number k such that the kth digit d_k in the decimal representation of the real number $(\pi+e)^{\pi-e}$ is the first of a series of 100 digits, all equal to d_k, then $a=\frac{1}{(-10)^n}$; if no such number k should exist, then $a=0$;

if n is the smallest number k such that the kth digit d'_k in the decimal representation of the real number $(\pi - e)^{\pi+e}$ is the first of a series of 100 digits, all equal to d'_k, then $b = \frac{1}{(-10)^n}$; if no such number k should exist, then $b = 0$.

There is at present no method which enables us to decide which of the three cases (i), (ij), or (iij) is realised.

It should be noted that, in other cases, equations may behave quite normally although their coefficients are defined in a similarly complex manner. Let $a' = a + 1$ and $b' = b + 1$ be the coefficients in the equation

$$a' \cdot x + b' = 0.$$

This equation has a unique solution x, and we can immediately state that:

$$-\frac{101}{90} \leqq x \leqq -\frac{90}{101}.$$

We meet with difficulties of a similar nature in elementary geometry.

It should be mentioned that these difficulties vanish when coefficients and variables are restricted to values in the field of real algebraic numbers; in this manner, almost the whole of elementary geometry can be protected against intuitionistic criticism (E. W. Beth, 1935). This result is related to Tarski's solution of the decision problem for elementary algebra and geometry (1939).

138. The Theory of Continuum

Such examples as have been given in the preceding sections will scarcely be sufficient to convince everyone of the intrinsic strength of the intuitionistic attitude.

Let us consider, for instance, the point of view of a platonist, whom we suppose to argue as follows:

My own ignorance in mathematics and my own incapacity to solve certain mathematical problems force me to admit the existence of a Supreme Intellect, for which no unsolvable problems can exist. To this Supreme Intellect, the difficulties indicated by Brouwer do not present themselves. So It will be aware which of the suppositions (i)–(iii) of Section 137 is true.

Now let us suppose that some conclusion C follows from any of these suppositions. Let us consider the three derivations starting from (i), (ii), and (iii) and all yielding the conclusion C. The Supreme Intellect knows which of the suppositions is true and therefore It also knows which of the three derivations constitutes a valid proof. It must, therefore, know the conclusion C to follow, by a conclusive derivation, from a true statement and therefore It knows the conclusion C to be true.

Now the disjunctive statement

$$(a = 0 \ \& \ b \neq 0) \vee (a = 0 \ \& \ b = 0) \vee a \neq 0$$

is in this situation. Therefore, though it is not considered a valid theorem by intuitionists, it can safely be accepted as true.

What objections can the intuitionist raise against this argument?

Let us first observe that the platonist's point of view is proof against a *reductio ad absurdum*. Such a *reductio* would, indeed, lead to a proof of the negation of the disjunctive statement discussed above, namely of the statement

$$(a \neq 0 \vee b = 0) \ \& \ (a \neq 0 \vee b \neq 0) \ \& \ a = 0$$

This statement which is, of course, false can neither be proved in intuitionistic nor in classical mathematics. It follows that the intuitionist cannot appeal to a *reductio ad absurdum*.

However, he can retort as follows. The platonist's argument depends on the thesis of the existence of a Supreme Intellect. We need not question the truth of this thesis. But it can scarcely be taken to constitute a suitable basis for mathematical proof. So the argument will be rejected by the intuitionist on the strength of his thesis that mathematics should not depend on principles of a non-mathematical nature.

The intuitionist has, however, no reason to restrict himself to so purely negative an argument. If he wants to give conclusive evidence, not for the exclusive tenability of the intuitionistic conception of mathematics, but of its intrinsic importance, he can point to the edifice of intuitionistic mathematics which has been built alongside classical mathematics.

For, better than intuitionistic criticism of classical mathematics, intuitionistic mathematics itself reveals the spirit behind intuitionism, a spirit more anxious to construct than to demolish.

If we have given a somewhat detailed exposition of intuitionistic criticism, it was in order to make intuitionistic mathematics more accessible, and to reveal some of the typical presuppositions underlying classical mathematics.

The central place in intuitionistic mathematics is occupied by the *theory of continuum*. This theory is of a completely original character on account of the introduction of the notion of an *infinite sequence of arbitrary choices*.

I will set forth the elements of this theory in a simplified version, given by A. Heyting in 1931; we shall restrict ourselves to an exposition of the construction of the closed linear continuum $C(0, 1)$.

We start from the following arrangement of dual fractions:

$$0,\ 1,\ \tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{3}{4},\ \tfrac{1}{8},\ \tfrac{3}{8},\ \tfrac{5}{8},\ \tfrac{7}{8},\ \tfrac{1}{16},\ \tfrac{3}{16},\ \ldots$$

A real number is generated by assigning to every dual binary fraction, encountered on going through this arrangement, one of the predicates *left* and *right*. These predicates can be chosen in an arbitrary manner, but the natural order in the system of rational numbers should be respected; that is, if some fraction f has been assigned the predicate *left*, then every fraction smaller than f must be given the predicate *left* as well, and if some fraction f' has been assigned the predicate *right*, then every fraction larger than f' must be given the predicate *right*. At each stage in the assignment procedure, the predicate to be assigned to one single fraction may be left indeterminate; as long as this predicate has not been chosen, the predicate to be assigned to the following fractions will be determined uniquely by the natural order existing in the system of rational numbers.

For instance, if we leave indeterminate the predicate of the fraction $\frac{1}{2}$, I am constrained to assign to $\frac{1}{4}$ the predicate *left*, to $\frac{3}{4}$ the predicate *right*. If I now decide to assign the predicate *left* to $\frac{1}{2}$, then the choice of the predicate to be assigned to $\frac{5}{8}$ will be free; and so on.

Let us consider a few examples to illustrate the procedure.

(1) The predicate to be assigned to $\frac{1}{2}$ is left indeterminate throughout the process; then every fraction $f<\frac{1}{2}$ will be given the predicate *left*, and every fraction $f'>\frac{1}{2}$ will be given the predicate *right*. The real number which is generated in this manner will be called *the real number* $\frac{1}{2}$.

(2) The fractions $f\leqq\frac{1}{2}$ are given the predicate *left*, the fractions $f'>\frac{1}{2}$ are given the predicate *right*. Except for the predicate assigned to the fraction $\frac{1}{2}$, this sequence of predicates coincides with the sequence

described under (1). We say, in such a case, that the two sequences *generate the same real number;* in our case, both sequences generate the real number $\frac{1}{2}$.

(3) The fractions $f<\frac{3}{5}$ are given the predicate *left*, the fractions $f'>\frac{3}{5}$ are given the predicate *right*. This choice sequence generates the real number $\frac{3}{5}$.

(4) The fractions f such that $f^2<\frac{1}{2}$ are given the predicate *left*, the fractions f' such that $f'^2>\frac{1}{2}$ are given the predicate *right*. This choice sequence generates the real number $\frac{1}{2}\cdot\sqrt{2}$.

We can now introduce arithmetical operations on real numbers.

(5) *Multiplication* of real numbers can be defined in the following manner. Let a and b be any real numbers. Suppose that the choice sequence which generates the real number a is already completed as to the choice of the predicates for the binary fractions with denominator $\leqq 2^n$, the choice of one of these predicates being possibly left indeterminate; let us rather say, more briefly: *the nth phase in generating the real number a has been completed.* Let the nth phase in generating the real number b also be completed.

We assume that, in the first choice sequence, $p/2^n$ has been given the predicate *left*, while $(p+2)/2^n$ has been given the predicate *right*. Likewise, let in the second choice sequence $q/2^n$ have the predicate *left*, while $(q+2)/2^n$ has the predicate *right*.

Now we introduce a third choice sequence by stating that, for every n, $p\cdot q/2^{2n}$ will have the predicate *left*, while $(p+2)\cdot(q+2)/2^{2n}$ will have the predicate *right*.

We can easily show that, in this manner, a third real number c is generated. Indeed, the difference between $p\cdot q/2^{2n}$ and $(p+2)\cdot(q+2)/2^{2n}$ amounts to $(2p+2q+4)/2^{2n}\leqq(\frac{1}{2})^{n-2}$. It follows that all but at most one of the binary fractions with denominator 2^{n-1} have been assigned a predicate. In other words: the $(n-1)$st phase in generating a certain real number c has been completed.

The real number c will be called the *product* $a\cdot b$ of the real numbers a and b.

(6) It is, of course, not possible to define arithmetical *addition* within the closed linear continuum $C(0, 1)$. We can however, in an obvious manner, introduce the *arithmetic mean* of two real numbers.

Example 1. Show that for the multiplication of real numbers, as defined above, we have the associative law:

$$a\cdot(b\cdot c) = (a\cdot b)\cdot c.$$

Example 2. Discuss the introduction of the arithmetic mean of two real numbers.

139. Intuitionistic Set Theory

In our general definition of the notion of a real number [belonging to the closed linear continuum $C(0, 1)$], we have restricted the liberty of choice only by introducing the condition that the natural order among rational numbers must be respected. On the other hand we have described in our examples (1)–(4), choice sequences in which there remained no liberty of choice whatsoever. It will be clear that, if we wish to be able to characterise a specific real number, such a description of the corresponding choice sequences is indispensable.

There remains, of course, a third possibility: we may restrict the liberty of choice by imposing on the choice sequences a certain condition which, nevertheless, leaves open, at least in certain phases, the choice of the predicates to be assigned to certain binary fractions.

A restrictive condition of this kind must be stated in such a manner that, for each finite sequence of permissible choices, at least one continuation of the sequence is permitted.

Suppose, for instance, that the choice of the predicates is restricted by the condition that $\frac{1}{2}$ must have the predicate *left*, while $\frac{3}{4}$ must have the predicate *right*. This condition does not affect the liberty of choice with regard to the predicates for the fractions $\frac{5}{8}$, $\frac{9}{16}$, $\frac{11}{16}$, $\frac{17}{32}$, $\frac{19}{32}$, $\frac{21}{32}$, $\frac{23}{32}$, …; it will be clear that this condition may be taken to constitute a definition of a certain *set of real numbers*, namely, for the closed interval $C(\frac{1}{2}, \frac{3}{4})$.

Likewise, we can define the closed interval $C(\frac{1}{3}, \frac{3}{5})$ by stating the condition that every binary fraction $f < \frac{1}{3}$ must have the predicate *left* and that every binary fraction $f' > \frac{3}{5}$ must have the predicate *right*.

If a and b are arbitrary real numbers, then the closed interval $C(a, b)$ can be defined by stating that every binary fraction f which in generating a has been given the predicate *left* must retain this predicate, while every binary fraction f' which in generating b has been given the predicate *right*, must also retain this predicate.

It may be felt that the acceptance of the choice sequence as a legitimate method of mathematical construction introduces into mathematics an element of subjectivity which has always been considered as utterly foreign to the field. However, this element of subjectivity can be eliminated by taking into account, in mathematical discussions, only those properties of real numbers and of sets of real numbers which do not depend on the choice sequences by which real numbers are actually generated but only on the specific conditions which have been imposed upon these choice sequences.

(1) We shall now show that, whenever the sets S and S' are *Brouwerian sets* (or "*spreads*"), that is, sets of real numbers defined by conditions C and C' of the kind which we have just described, their *union* $S \cup S'$ will also be a Brouwerian set. We shall, in fact, state a condition C'' which is of the same kind and which constitutes a definition of the set $S \cup S'$.

Suppose that the nth phase in generating some real number x intended to be in $S \cup S'$ has been completed. The choices of the predicates must be either (i) in accordance with condition C, or (ij) in accordance with condition C', or (iij) in accordance with both condition C and condition C'; in case (i), the $(n+1)$st choice must be made in accordance with condition C; in case (ij), in accordance with condition C'; and in case (iij), either in accordance with condition C or in accordance with condition C'; in any case, it will be possible to continue the series.

(2) For the intersection $S \cap S'$ of two Brouwerian sets S and S', and

(3) For the complement $\bar{S}$ of a Brouwerian set S, we cannot give a similar argument. It follows that $S \cap S'$ and $\bar{S}$ cannot, in general, be expected to constitute Brouwerian sets.

(4) The limitations inherent in set-theoretic operations on Brouwerian sets have led Brouwer to introduce, in addition to the notion of a Brouwerian set (or spread), the notion of a *species*, which is considerably larger and comes nearer to the classical conception of a set or class. A species is characterised by a specific property of its elements. Thus we can consider the species of the real numbers which are elements of a given Brouwerian set S; the species of the real numbers which are not elements of a given Brouwerian set; etc.

The complement of a species and both union and intersection of two species are species. Hence, the subspecies of a given set or species (for instance, the closed linear continuum $C(0, 1)$) constitute a lattice. However, they do not constitute, intuitionistically, a Boolean algebra. Indeed, set-theoretic operations on species are also affected by complications which are unknown in classical mathematics; this is apparent from the following theorem.

(5) Suppose that to every real number on the closed linear continuum $C(0, 1)$ we have assigned one of the predicates u or v; let some real number a on $C(0, 1)$ have the predicate u; then *every* real number b on $C(0, 1)$ must have the predicate u.

Proof. Suppose that some real number b on $C(0, 1)$ has the predicate v; let m_1 be the arithmetic mean of a and b; there will be two cases:

(i) m_1 has the predicate v; in this case, we take $a_1 = a$, $b_1 = m_1$;
(ij) m_1 has the predicate u; then we take $a_1 = m_1$, $b_1 = b$.

By iterating this procedure, we clearly obtain a sequence of real numbers $a_1, a_2, \ldots, a_k, \ldots$, all having the predicate u, and a sequence of real numbers $b_1, b_2, \ldots, b_k, \ldots$, all having the predicate v.

Let us consider the kth phase in generating each of the numbers $a_k, a_{k+1}, a_{k+2}, \ldots, b_k, b_{k+1}, b_{k+2}, \ldots$. It is easy to see that, except for at most one predicate, the corresponding finite choice sequences must agree. Hence we can define a real number m on $C(0, 1)$ such that, for every k, the kth phase in generating m presents exactly those predicates as to which the above choice sequences happen to agree.

Now let a real number n be generated as follows. We start choosing the predicates in accordance with the choice sequence for m, but we make the mental reservation that, in any phase, we shall have the right to change over to the choice sequence either for the corresponding number a_k or for the corresponding number b_k. Thus, throughout the process of generating the real number n, we can indefinitely postpone the decision as to the predicate u or v to be assigned to n. But this is clearly inconsistent with our supposition that to every real number on $C(0, 1)$ one of the predicates u and v has already been assigned.

So we have refuted the supposition that some number b has the predicate v; and it follows that every real number on $C(0, 1)$ has the predicate u.

(6) It follows that the complement $\bar{S}$ of a subspecies S of $C(0, 1)$ can only exist in the trivial case that either S or $\bar{S}$ is empty.

140. Brouwer's Fundamental Theorem on Finitary Spreads

It will now be convenient to state spread laws in a slightly different form (the spread laws discussed so far can be restated accordingly).

(1) The sequence of binary fractions is replaced by the sequence of all non-negative integers:

$$0, 1, 2, \ldots, p, \ldots,$$

and, instead of assigning to each binary fraction f a certain predicate (or leaving it indeterminate), we assign to each non-negative integer p a certain natural number n_p. On account of a given spread law, some choices for n_p will be *sterile*; such a choice *sterilises* the choice sequence $n(p+1) = [n_0, n_1, n_2, \ldots, n_p]$, that is, it prevents its continuation and destroys the effect of the preceding non-sterile choices $n_0, n_1, n_2, \ldots, n_{p-1}$. However, if the sequence $n(p+1)$ is not sterilised, then at least one non-sterile choice for n_{p+1} must be available; what choices are available for n_{p+1} will, in general, depend upon the preceding choices. We shall say that, by making a non-sterile choice for n_{p+1} we *generate* the sequence $n(p+2)$.

(2) Those infinite sequences $n = [n_0, n_1, \ldots, n_p, \ldots]$ which are generated in accordance with the spread law are the elements of a species N, which is called a *spread*.

The finite sequences $n(p+1) = [n_0, n_1, \ldots, n_p]$ for $p = 0, 1, 2, \ldots$ are called the *initial segments* of n.

(3) $M = \bigvee_p E_{n(p)}$ will be the species of *all* initial segments, sterile or non-sterile.

$M(m(q)) = \bigvee_{p>q} E_{n(p)}[n_0 = m_0, n_1 = m_1, \ldots, n_{q-1} = m_{q-1}]$ will be the species of all initial segments $n(p)$ which are *extensions* of a given initial segment $m(q)$.

The species $N = E_n$[for any p, $n(p)$ is non-sterile] coincides with the spread N as defined above.

$N(m(q)) = E_n[n \in N, n_0 = m_0, n_1 = m_1, \ldots, n_{q-1} = m_{q-1}]$ is the species of all elements n of N which have a given initial segment $m(q)$.

It is easy to see that $N(m(q))$ is not only a subspecies, but even a sub-spread, of N.

(4) In this connection it should be pointed out that a spread N can be empty. The spread N will be empty if and only if, already for n_0, no non-sterile choice is available. For, if some choice for n_0 is non-sterile, then $n(1)=[n_0]$ is not sterilised, and hence a non-sterile choice for n_1 must be available, and so on.

(5) Now let us suppose we are given a function F which assigns, to each element n of N, a natural number $b=F(n)$ as its value. The definition of such a function F must involve the description of a certain algorithm which enables us to calculate, in a finite number of steps, each of the values $F(n)$. It will be clear that in such a calculation only the choice of some initial segment $n(p)$ of n can actually play a rôle; it follows that all elements n' in $N(n(p))$ must yield the same value $F(n')=F(n)$.

This consideration suggests the introduction of the following definitions, which are all relative to one particular spread N and to one particular function F defined on N.

(6) M_1 will be the species of all initial segments $m(q)$ such that the algorithm provides the value $F(n)$ for every n in $N(m(q))$ as soon as $n_q=m_q$ has been chosen, but not earlier.

M_2 will be the species of all initial segments $n(p)$ which satisfy one of the following conditions:

(i) $n(p)$ is sterile;

(ij) there is an initial segment $m(q)$ such that $n(p)$ is in $M(m(q))$ and $m(q)$ is in M_1.

If $n(p)$ is in M_2 then either $N(n(p))$ is empty, or $F(n)$ has already been decided by the choice of n_q for some $q<p$; hence, for all elements n' in $N(n(p))$, $F(n')$ must take the value $F(n)$.

M_3 will be the species of all initial segments $n(p)$ such that, for any $n'(p')$ in $M(n(p))$, we can find an initial segment $n''(p'')$ which satisfies the following conditions:

(i) $n''(p'')$ is in $M(n'(p'))$;

(ij) $n''(p'')$ is in M_2.

(7) Now suppose that, for some initial segment $n(p)$, we have a proof of the assertion:

$$n(p) \in M_3.$$

Then it must be possible to state this proof in such a manner that only the following schemes of derivation are applied:

$$\text{(I)} \qquad \frac{n'(p') \in M_2}{n'(p') \in M_3} \qquad\qquad \text{(II)} \qquad \frac{\begin{array}{c}[n'(p'), 1] \in M_3 \\ [n'(p'), 2] \in M_3 \\ \dots\dots\dots \\ [n'(p'), k] \in M_3 \\ \dots\dots\dots \end{array}}{n'(p') \in M_3}$$

[We have introduced a new notation, writing "$[n(p), n_p]$" instead of "$n(p+1)$".] Each assertion "$n'(p') \in M_3$" appears at most once as a premiss and at most once as a conclusion. Therefore, we may apply the method of definition by transfinite recursion on the derivation to introduce a decomposition of the sub-spread $N(n(p))$ of N into a well-ordered species $S(n(p))$ of sub-spreads $N(m(q))$ of N; as follows:

(i) If $n'(p')$ is in M_2, then $S(n'(p'))$ will be $\{N(n'(p'))\}$;

(ij) If, for $k=1, 2, \dots, S([n'(p'), k])$ is given, then $S(n'(p'))$ will be the ordinal sum:

$S([n'(p'), 1])+S([n'(p'), 2])+ \dots +S([n'(p'), k])+ \dots.$

For each element $N(m(q))$ of $S(n(p))$, $m(q)$ is in M_2, and the union of all these spreads $N(m(q))$ will be $N(n(p))$.

(8) Now the definition of the function F must be given in such a manner that we are able to prove that the initial segment $n(0)$ is in M_3. For, unless such a proof is given, we cannot be sure that the value $F(n)$ can be effectively calculated for each element n of N.

It follows that $N(n(0))=N$ is decomposable into a well-ordered species $S(n(0))$ of sub-spreads $N(m(q))$, such that $m(q)$ is in M_2 and that in $N(m(q))$ the value of F is constant.

(9) On the other hand, a *finitary spread* N is characterised by the fact that, for any non-sterilised initial segment $n(p)$, the non-sterile choices available for n_p are finite in number. It follows then, by transfinite recursion on the derivation, that $S(n(0))$ can only contain a finite number of non-empty subspreads $N(m(q))$. Hence we can effectively calculate the largest number z such that $N(m(z))$ is not

empty. It follows that, for any element n of N, the value $F(n)$ can only depend upon the choice of the initial segment $n(z)$.

(10) So we have Brouwer's *Fundamental Theorem on Finitary Spreads* (1923):

Suppose that on a finitary spread N a function F is defined, the values $F(n)$ of which are natural numbers; then there is a number z such that, for any element n of N, the value $F(n)$ depends only upon the choice of the initial segment $n(z)$.

(11) In an entirely similar manner, we can prove a theorem which is slightly more general:

Suppose that a finitary spread N is covered by a denumerable sequence of species $S_1, S_2, \ldots, S_k, \ldots$; then there is a number z such that, for any element n of N, there is a species S_k in which n is contained and which depends only upon the choice of the initial segment $n(z)$.

(12) Returning to the functions F defined on a finitary spread N, we can restrict ourselves to considering those functions F which take only two different values, say, 0 and 2. It will be clear that the species $E_n[F(n)=2]$ will constitute a denumerable Boolean algebra. It would be interesting to know:

(i) in which manner the structure of this Boolean algebra $B(N)$ depends upon the choice of N;

(ij) under which conditions the Boolean algebras $B(N)$ and $B(N')$ corresponding to two different spreads N and N' are isomorphic;

(iij) under which conditions a given denumerable Boolean algebra is isomorphic to a Boolean algebra $B(N)$.

Perhaps the work of J. C. E. Dekker (1953) will provide some results in this direction.

(13) The species $E_n[F(n)=2]$ on a finitary spread N play, to some extent, the rôle of open-closed subsets on a topological space T. Such a space T is always of dimension 0, and it follows that such a space as the closed linear continuum $C(0, 1)$ cannot be obtained directly as a finitary spread. Nevertheless, the construction of compact topological spaces can be carried out starting from the construction of suitable finitary spreads. An exhaustive treatment of this subject has been given by H. Freudenthal (1937).

(14) I have to restrict myself to restating our previous construction of $C(0, 1)$. We may write "1", "2", and "3" instead of "left", "right", and "left indeterminate". Then the sequence of predicates obtained at completing the pth phase in generating a real number a on $C(0, 1)$ is replaced by a sequence of (2^p+1) numerals "1", "2", and "3". Let n_p be the natural number denoted by this sequence of numerals. Then our description of the manner in which the real number a is generated may be restated in the form of an instruction concerning the non-sterile choice of a number n_p to extend a non-sterilised initial segment $n(p)$; this instruction may be construed as a spread law, and this spread law defines a certain finitary spread N. The actual wording of the spread law is not relevant.

(15) In order to conform to current usage, I adopt a terminology which is slightly different from Brouwer's. The elements of N will be called *generating elements* (instead of *points*). Two generating elements will be called *congruent*, if the corresponding sequences of predicates agree except for at most one single predicate.

A species of mutually congruent generating elements will be called a *point* (instead of a *point nucleus*), and the species of all points will be the closed linear continuum $C(0, 1)$.

The species $E_n[F(n)=2]$ on N will be called *species of first order*. We observe that, for a point P on $C(0, 1)$, there will, in general, be an infinite species $g(P)$ of generating elements in N.

(16) Now let us suppose that $C(0, 1)$ is covered by a sequence $T_1, T_2, \ldots, T_k, \ldots$ of species. Will there always be a number z similar to the one described under (11)?

Of course, it can be said that there must be a certain algorithm which, for any point P and for any generating element n in $g(P)$, provides a number k and an initial segment $n(p)$ such that P itself as well as any point P' which has a generating element n' in $N(n(p))$ is contained in T_k. However, the number k and the initial segment $n(p)$ may depend upon the element n in $g(P)$ with which we start and, moreover, if both n and n^* are in $g(P)$ then P' may have a generating element n' in some $N(n(p))$ without having any generating element in any spread $N(n^*(p^*))$. At this point, our previous argument breaks down.

(17) However, let T be a subspecies of $C(0, 1)$ which has the following property: there is a corresponding subspecies $S(T)$ of N

such that, if P is in T, then $g(P)$ is included in $S(T)$, and if P is not in T, then $g(P)$ is included in $N-S(T)$. Then T is called an *open subspecies* of $C(0, 1)$.

(18) On the basis of this definition, we can now prove the following *compactness theorem for the closed linear continuum* $C(0, 1)$:

Suppose that the closed linear continuum $C(0, 1)$ *is covered by a sequence* $T_1, T_2, \ldots, T_k, \ldots$ *of open subspecies; then there is a number* z *such that, for every point* P *on* $C(0, 1)$, *there is a species* T_k *in which* P *is contained and which depends only on the choice of the initial segment* $n(z)$ *for an arbitrary generating element* n *in* $g(P)$.

(19) This result provides the background for the somewhat startling theorem (5) in Section 139. Suppose we had covered $C(0, 1)$ by two disjoint species T_1 and T_2. Then we could define two subspecies S_1 and S_2 of N, as follows:

(i) n is in S_1, if and only if we can find a point P on $C(0, 1)$ such that n is in $g(P)$ and P is in T_1;

(ij) n is in S_2, if and only if we can find a point P on $C(0, 1)$ such that n is in $g(P)$ and P is in T_2.

Now let n be any element of the spread N. The species of all elements which are congruent with n determines a certain point P on $C(0, 1)$, and we know, of course, that n must be in $g(P)$. Now if P is in T_1, then n is in S_1 and so is any other element in $g(P)$, hence $g(P)$ is included in S_1; moreover, no point P' can be found such that P' is in T_2 and n, or any other element n' in $g(P)$, is in $g(P')$, so n is in $N-S_2$ and $g(P)$ is included in $N-S_2$.

Similarly, if P is in T_2, then n is in S_2 and in $N-S_1$, and $g(P)$ is included both in S_2 and in $N-S_1$.

Moreover, $C(0, 1)$ is covered by T_1 and T_2, so P must be either in T_1 or in T_2.

It follows that both T_1 and T_2 must be open species. But it is known in classical mathematics that, if $C(0, 1)$ is covered by two disjoint open point sets, then either of the two sets is empty.

(20) The fact that in intuitionistic mathematics a compactness theorem such as theorem (18) can be proved is interesting. In classical mathematics, many existence theorems are either themselves compactness theorems or they depend upon compactness theorems. As

an illustration we mention the theorems concerning the existence of models for certain classes of deductive theories, which could be interpreted as compactness theorems for certain topological spaces. Another example is found in a result of J. L. Kelley (1950), according to which Tychonoff's theorem concerning the compactness of every Cartesian product of compact topological spaces entails the axiom of choice.

This close connection between compactness theorems and existence theorems now turns out to subsist if from classical mathematics we pass on to intuitionistic mathematics.

I shall later return to the discussion of certain critical points in Brouwer's proof of the Fundamental Theorem on Finitary Spreads.

But it will be clear by now that justice is not done to intuitionistic mathematics when it is described as the fragment of classical mathematics which results from the elimination of those elements which do not sustain intuitionistic criticism. On the contrary, Brouwer has introduced new and original methods, which have no counterpart – or, at least, no obvious counterpart – in classical mathematics. After Brouwer had laid the foundations of intuitionistic mathematics and established its basic theories, the subject was further developed by M. Belinfante (infinite progressions, 1929, 1930, 1938; functions of a complex variable, 1938, 1941), J. G. Dijkman (infinite progressions, 1946, 1948), H. Freudenthal (topology, 1936), A. Heyting (projective geometry, 1925; algebra, 1943; theory of Hilbert space, 1951), B. de Loor (theorem of d'Alembert, 1928), B. van Rootselaar (measure and integration, 1954). Of course, we are not concerned here with these developments, but rather with the foundations of intuitionistic mathematics and with its connections with classical mathematics.

141. Heyting's Formalisation of Intuitionistic Logic (1930)

A systematic treatment of intuitionistic logic was given by Heyting in the context of a formalisation of intuitionistic mathematics. Previously, Brouwer (1925), A. N. Kolmogorov (1925), A. Khintchine (1928), and V. Glivenko (1929) had already studied various special subjects in intuitionistic logic. Heyting's vast enterprise was elicited by the polemics of M. Barzin and A. Errera (1927–1933), and it gave rise to a number of important publications.

It should, perhaps, be emphasised once again that, for an intuitionist, no formalisation can constitute a foundation for intuitionistic mathematics; it can give no more than a basically inadequate image of it. Hence the divergences existing between formalisations of intuitionistic, and formalisations of classical, mathematics are only of secondary importance; the main difference is between the attitudes adopted, by intuitionists and by adherents of classical mathematics, in the interpretation of mathematical theories, whether formalised or not. We shall see (in Sections 142 and 143) that even among intuitionists there is no complete agreement as to the interpretation of intuitionistic logic, as formalised by Heyting. However, such differences of opinion do not constitute an objection to intuitionism; they rather prove that a clarification has resulted from Heyting's work.

It is emphasised by Heyting that intuitionistic mathematics is not dependent on the existence of logical principles of universal validity. On the contrary, the validity of a logical principle must be ascertained every time it is applied in a mathematical proof.

Nevertheless, Heyting begins his construction of a formalisation for intuitionistic mathematics by enumerating a number of logical theses which can be safely applied in intuitionistic mathematics; this is done in order to retain, as far as possible, the analogy with extant formalisations of classical mathematics. – Later, G. Gentzen (1934), S. Jaskowski (1934), and others developed new methods which afford more "natural" formalisations of deductive theories. These methods, which give preference to inference schemes at the expense of logical theses, allow us to make a comparison between intuitionistic and classical logic in a very elegant manner; we shall return to this point in Section 145.

We have already seen (in Section 135) that in intuitionistic logic neither the principle of the excluded third:

(a) $$p \vee \bar{p}$$

nor the thesis:

(b) $$(x)a(x) \vee (Ex)\overline{a(x)}$$

of classical logic can be accepted as universally valid principles. In classical logic, we can derive (b) from (a) on account of the validity of a third thesis:

(c) $$\overline{(x)a(x)} \rightarrow (Ex)\overline{a(x)}$$

which, however, also fails to be universally valid in intuitionistic logic. Therefore, the theory of quantification in intuitionistic logic presents complications unknown in classical logic. – (Of course, the rejection of logical theses such as (a), (b), and (c) in intuitionistic logic does not imply the acceptance of their negations as universally valid; in fact, these negations are not even accepted as sometimes valid.) –

On account of the peculiarities of intuitionistic logic, the elimination of quantifiers, which must be effected in order to obtain solutions of certain special cases of the decision problem, is not always permissible. However, in those cases in which an effective solution holds classically, this solution is sometimes also acceptable intuitionistically, and then the necessary reductions receive a justification *post factum*.

For instance, Tarski's solution of the decision problem for elementary algebra and geometry (1938, 1948) is acceptable from an intuitionistic point of view, provided only that the range of the variables be restricted to algebraic real numbers. Independent of Tarski's result, the present author (1935) observed that this fact entails an intuitionistic justification of the more elementary parts of geometry.

To conclude, I briefly mention a number of metalogical results (some more are contained in the Examples at the end of Section 145).

Gödel (1933) observed that the intuitionistic sentential calculus cannot have a finite characteristic matrix. By Lindenbaum's theorem (*cf.* Section 88) it must at any rate have an infinite characteristic matrix, and such a matrix was effectively constructed by S. Jaskowski (1936). Moreover, Gödel conjectured a close connection between the intuitionistic and the modal sentential calculi.

M. H. Stone (1937) and A. Tarski (1938) revealed connections between the intuitionistic sentential calculus and the algebra of closed (or open) subsets on topological spaces; a similar connection was pointed out by J. C. C. McKinsey (1941) for the modal sentential calculus. These results were given an algebraic form by McKinsey and Tarski (1944, 1946, 1948); the connections between the intuitionistic and the modal sentential calculi were at the same time solidly established.

Mostowski (1948) and Henkin (1950) indicated means of extending the algebraic methods developed by Tarski and McKinsey to intuitionistic and modal predicate logic. Helena Rasiowa (1950) obtained, for these systems, results analogous to the theorems of Löwenheim, Skolem, and Gödel for classical predicate logic; more work in this direction has been done by Rasiowa (1951, 1952, 1954) and by Rasiowa and Sikorski (1953, 1954). On the other hand, S. C. Kleene (1945, 1948, 1952), D. Nelson (1944, 1947), and G. Rose (1952) applied the theory of recursive functions to give an interpretation of intuitionistic logic and mathematics. These investigations are somehow connected with those discussed in Section 145, but they are less closely connected with the fundamental conceptions underlying intuitionistic mathematics.

Gödel's work on intuitionistic arithmetic afforded a starting-point for studies by D. van Dantzig and G. F. C. Griss, which will be discussed in Section 142. It should be mentioned that the main ideas underlying these studies have been stated independently by Bernays at the *Entretiens de Zurich* (1938, published 1941). In this connection, I wish also to mention work by G. Mannoury (1925, 1943) and by A. Reymond (1936) and, in particular, the construction by I. Johansson (1936) of the *minimal calculus*, a system of sentential calculus still weaker than the intuitionistic calculus, and having rather peculiar metalogical properties.

142. Van Dantzig's Stable and Affirmative Mathematics (1942, 1947) – Griss's Negationless Intuitionistic Mathematics (1944, 1946, 1950)

The principal aim of van Dantzig's investigations is to fill the gap which still exists between classical and intuitionistic mathematics. In order to realise this purpose he applies two divergent methods.

(1) The first method consists in extending Gödel's result concerning the connections between classical and intuitionistic arithmetic to an essential and, if possible, extensive fragment of analysis. Since we can, in classical analysis, distinguish a more elementary part, which is independent of set theory – and especially of the axiom of choice and of the continuum hypothesis –, and a more advanced part, in which methods borrowed from set theory play an important rôle, it seems reasonable to conjecture that Gödel's result still applies to the more elementary part. Such an extension might even be expected to result immediately when the formal procedure, mentioned above, is applied to the theorems of elementary classical analysis. In this manner, every theorem is replaced by a stable expression, that is,

by an expression which is – classically as well as intuitionistically – equivalent to its double negation; the classical proof of the original theorem is replaced by an intuitionistic proof of its stable counterpart.

Simple and obvious though this procedure may appear, it meets with a serious difficulty: in our proofs we must sometimes refer to definitions. Now the definitions which are introduced in order to pass from arithmetic to analysis (*cf.* Sections 38–40) are not usually given a stable form. Therefore, a revision of these definitions is indispensable; such a revision has been carried out by van Dantzig for a number of fundamental notions of elementary analysis.

We can characterise van Dantzig's stable mathematics as an attempt to develop a fragment of intuitionistic mathematics which should reproduce literally an elementary fragment of classical analysis.

(2) The second method consists in the establishment of a fragment of elementary classical analysis on the basis of suppositions as weak and obvious as possible. Moreover, van Dantzig avoids the introduction of those logical operators wich give rise to complications in intuitionistic mathematics, namely, negation, disjunction and the existential quantifier. In this respect van Dantzig's affirmative mathematics recalls similar attempts by Carnap (1934), Church (1936), and Quine and Goodman. Van Dantzig himself quotes I. Johannson (1936).

Though van Dantzig's studies present many interesting features and may be said, in fact, to constitute a valuable contribution to a better understanding of the foundations of classical analysis, I do not think that his methods can be expected to yield a reconstruction of anything more than an elementary and logically weak part of classical analysis.

While van Dantzig attempts to fill the gap between classical analysis and extant intuitionistic mathematics, G. F. C. Griss (1944, 1946, 1950) presents a programme which is even more radical than Brouwer's views.

Not content with eliminating negation – the absence of which is partly compensated by admitting, in addition to identity, distinctness as an undefined relation –, Griss also rejects disjunction and the introduction, in mathematical proofs, of suppositions not previously realised (for a definition of distinctness, *cf.* p. 672).

The starting-point of negationless intuitionistic mathematics is the construction of the series of natural numbers; it is stated, as intuitively obvious, that each natural number is distinct from every preceding one. On the basis of this construction a comparatively extensive part of extant intuitionistic mathematics can indeed be reconstructed.

Of course, part of extant intuitionistic mathematics is absent in Griss's reconstruction and, for the remaining part, a thorough revision is necessary.

However, to the present author, Griss's reconstruction seems not to be satisfactory in all respects.

(1) Though, in general, the use of disjunction as a sentential connective is not admitted, it is used in special contexts on the basis of the following definition:

"*a* or *b* is true for all elements of the set V means that the property *a* holds for a subspecies V' and the property *b* holds for a subspecies V'', V being the sum of V' and V''."

It seems to me that, in accordance with the conventions introduced in Section 88, this definition should be read as follows:

"*a* or *b* holds for all elements of the set V" means that the condition *a* holds for all elements of a subspecies V' and condition *b* holds for all elements of a subspecies V'', such that V is the sum of V' and V''.

It seems clear that, although *a* and *b* are, at first, stated to be sentences (assertions), they are actually used as conditions (or sentential functions), containing some variable x ranging over the elements of a certain species V. It follows that the definition presents no real interest, as it only affords a method for defining the disjunction of two sentential functions in terms of the sum or union of two subspecies of a given species. Moreover, I feel that, from a constructive point of view, there are certain objections to admitting addition of species as an undefined operation.

For instance, I have suggested the following definitions of the *disjunction* of sentences or sentential functions p and q:

$$p \vee q \leftrightarrow (x)(y)[(z)\{(p \to z \# y) \; \& \; (q \to z \# y)\} \to y \# x],$$

and of the *union* of species A and B:

$$x \in [A \cup B] \leftrightarrow (y)[(z)\{(z \in A \to z \# y) \; \& \; (z \in B \to z \# y)\} \to y \# x];$$

However, according to P. C. Gilmore (1953), these definitions are not suited to our purpose.

(2) Griss rejects *empty species*, which again causes considerable complications in the establishment of a calculus of species. Multiplication of two given species is allowed only, if a common member of these species has been constructed previously. Now this complication is superfluous; it can be eliminated in the following manner. Suppose that we wish to construct a calculus of species A, B, C, ... of natural numbers. Then, before constructing the series of natural numbers, we first construct another element, say z, which, for the time being, is held apart. The construction of the series of natural numbers is then carried out.

We now introduce a species $\emptyset'$ which only contains the element z. Moreover, we replace every species A of natural numbers which is recognised by Griss by a species A', containing every natural number in A and, also, the element z. Now we can, in dealing with the species $\emptyset'$, A', B', C', ..., apply multiplication without any restriction.

On the other hand, the resulting calculus of species is, obviously, the calculus which we should have obtained if, besides the species A, B, C, ... of natural numbers, an empty species had been introduced from the beginning.

Negationless axiomatics has been studied, since the publication of Griss's first papers, by Paulette Destouches–Février (1947, 1948, 1949), R. de Bengy–Puyvallée (1947), Nicole Dequoy (1949), P. G. J. Vredenduin (1953), and P. C. Gilmore (1953).

143. Brouwer's Comments on Essentially Negative Predicates (1948)

It is interesting to note that van Dantzig's and Griss's ideas, and more especially their attempts to eliminate negation from intuitionistic mathematics – this tendency probably derives from Mannoury's significs –, are not endorsed by Brouwer. As a matter of fact, the creator of intuitionistic mathematics has shown, in a series of notes (1948, 1949), that there are essentially negative predicates in intuitionistic mathematics, that is, predicates which can hardly be expected to be introduced without an appeal to negation; the following illustrative example has been occasionally presented by Brouwer in lectures and conferences since 1927. Let A be a mathematical assertion which cannot be *tested*, that is, for which there is no recognised method of deducing either its absurdity or the absurdity of its absurdity; for instance, A may be the assertion that, for some k, the kth digit in the decimal representation of the real number π is the first in a sequence of digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Then the creative subject can, in connection with the assertion A, create an indefinite sequence of rational numbers $a_1, a_2, a_3, \ldots$, in accordance with the following instructions:

(i) When, up to the moment of choosing a_n, neither the truth nor the absurdity of the assertion A has become evident to the creative subject, the choice will be $a_n = 0$.

(ii) When, between the choice of a_{r-1} and a_r, the creative subject has obtained conclusive evidence for the truth of A, then the subsequent choices will be $a_r = a_{r+1} = \ldots = a_{r+k} = \ldots = 2^{-r}$.

(iij) When, between the choice of a_{s-1} and a_s, the creative subject has obtained conclusive evidence for the absurdity of A, then the subsequent choices will be $a_s = a_{s+1} = \ldots = a_{s+k} = \ldots = -2^{-s}$.

The sequence $a_1, a_2, a_3, \ldots$ is positively convergent, and hence defines a real number a.

Now, if we had $a > 0$, then $a < 0$ would be absurd; so the absurdity of the absurdity of A would have been established, hence A would have been tested, contrary to our supposition. So $a > 0$ cannot be the case.

If we had $a < 0$, then $a > 0$ would be absurd; so the absurdity of A would have been established, hence A would have been tested, contrary to our supposition. So $a < 0$ also cannot be the case.

If we had $a = 0$, then both $a > 0$ and $a < 0$ would be excluded; so both the absurdity of A and the absurdity of the absurdity of A would have been established. Hence the supposition that $a = 0$ implies a formal contradiction; it follows that $a \neq 0$.

Consequently, for the numbers a and 0, we have $a \neq 0$, without either $a > 0$ or $a < 0$. As to the constructive order relations, which are stronger, we also cannot have either $a \circ\!> 0$ or $a <\!\circ\, 0$. So the predicate $\neq$ seems to be *essentially negative*, that is, not definable in terms of constructive predicates without an appeal to negation or absurdity.

If $b = |a|$, then we have $b > 0$ without $b \circ\!> 0$. So the predicate $>$ is also essentially negative. We shall not follow Brouwer in the further development of this train of thought, but shall rather turn our attention to van Dantzig's reply (1949).

Van Dantzig observes that Brouwer's terminology is not completely clear and lends itself both to an "*objectivistic*" and to a "*subjectivistic*" interpretation of his views; it is suggested – rightly, as it appears – that a subjectivistic interpretation would be closer to Brouwer's intentions; however, the validity of Brouwer's argument does not depend upon the choice which we make between an objectivistic and a subjectivistic interpretation.

What really matters is (a) the acceptance of absurdity as a basic notion in intuitionistic mathematics, and (b) the absence, in intuitionistic mathematics, of assertions which can be proved to be incapable of decision.

Since Brouwer's argument is based essentially on the acceptance of the notion of absurdity, van Dantzig rightly observes that it cannot prove anything against those attempts at a reconstruction of intuitionistic mathematics which have been presented by Griss, Paulette Destouches-Février, and by van Dantzig himself, and in which the introduction of the notion of absurdity is systematically avoided. On the contrary, this argument shows once again how unclear the notion of absurdity is, and how desirable it is to avoid its use. On the other hand, it must be granted that Brouwer's argument makes apparent the loss which a wholesale elimination of absurdity from intuitionistic mathematics would entail.

The absence from intuitionistic mathematics of assertions which can be proved to be incapable of decision makes it appear dubious whether it will be possible, in the long run, to maintain the strict distinction between formal systems and their metasystems against Brouwer's rejection of such a distinction. In this connection, the following critical remarks seem apposite. First, there are not only assertions incapable of decision in certain formal systems, there are also problems essentially incapable of decision. The proofs of the existence of such problems depend, as a matter of fact, on a distinction between certain formal systems and their corresponding metasystems. However, such a distinction seems unavoidable if reasonable standards of formal rigour are accepted; we have seen, moreover, that Brouwer himself, in his thesis, actually made a distinction of this nature. The acceptance of a constructivistic attitude seems to me to be consistent with a distinction between formal systems and their corresponding metasystems.

It should, once again, be emphasised that this distinction is a strict one, but by no means a rigid one. Suppose we are given a formal system S and a corresponding metasystem M; then there is no objection to the construction of a formal system S′ in which both S and M can be interpreted. However, the metasystem M′ which corresponds to the formal system S′ will be different from M. So the distinction depends entirely on the choice of a domain to be formalised and subjected to a metamathematical examination.

It may be added that the demand for a metamathematical analysis of existing theories of intuitionistic mathematics, made by Church (1939), seems completely justified; only such an analysis will be able to clear up the status of essentially unsolvable mathematical problems, as constructed by Church (1936), in relation to intuitionistic mathematics, which allegedly cannot recognise the existence of such problems.

144. Intuitionism and Semantics

It seems to me that the problems arising from the development of intuitionistic, stable, affirmative, and negationless intuitionistic mathematics would be very much clarified if a suitable adaptation of the method of semantics, as introduced by Tarski, should prove possible. I have already mentioned Kleene's and Nelson's application of this method to the problem of interpreting intuitionistic arithmetic.

In connection with the eventual application of this method in a logical analysis of intuitionistic set theory, I should mention an objection made in conversation by van Dantzig and Freudenthal. There are procedures, especially in the theory of finitary sets, which appeal to notions belonging essentially to metamathematics; this circumstance might turn out to render any attempt to effect a suitable separation between mathematics and metamathematics illusory with regard to this theory. Of course, the possibility of a semantical analysis of intuitionistic set theory depends essentially on the success of such an attempt.

In my opinion, however, there are sufficient reasons for not renouncing too quickly the application of the methods of semantics to a logical analysis of intuitionistic set theory.

(1) The method of the arithmetisation of metamathematics and the logico-mathematical parallelism allow the association of a considerable number of metamathematical notions and problems with notions and problems of a strictly mathematical nature. Of course,

it remains to be seen whether these devices can be adapted to cover the notions and problems which present themselves in intuitionistic set theory.

(2) It is possible to give a paraphrase of Brouwer's so-called set definition which is in complete agreement with the principles of semantics. This paraphrase takes the form, not of a definition of the notion of a set, but of a definition of the notion of a *set definition* or a *spread law.*

"A spread law or set definition is an instruction according to which, when we repeatedly choose an arbitrary natural number as an *index*, each of these choices has as its predeterminate effect (which may depend also on the preceding choices) that either a certain *figure* (that is, either nothing or some mathematical entity) is generated or the choice is *sterilised.* In the latter case the figures generated so far are destroyed and generation of any further figures is prevented, and hence every further choice will be likewise sterilised.

The only condition to be satisfied is that, after each non-sterilised sequence of $n-1>0$ choices, at least one natural number must be available which, if chosen as the nth index, effects the generation of a figure."

"The infinite sequences of figures generated in agreement with a spread law on account of indefinitely proceeding sequences of choices are, by virtue of this genesis, and together with any infinite sequences identical with one of them, the elements of a species. Such a species is called a *spread.*"

So every *spread law* or *set definition* may be given the following symbolical form:

$$\alpha_1 = M'_1, \qquad\qquad p_1 = M_1(\tau\alpha_1),$$
$$\alpha_{k+1} = M'_{k+1}(\tau\alpha_1, \tau\alpha_2, \ldots, \tau\alpha_k), \quad p_{k+1} = M_{k+1}(\tau\alpha_1, \tau\alpha_2, \ldots, \tau\alpha_{k+1}).$$

The relation $\in$ between a spread and its elements may then be defined as follows:

$$X \in M' \leftrightarrow x_1 \in M'_1 \;\&\; (k)[x_{k+1} \in M'_{k+1}(x_1, x_2, \ldots, x_k)],$$
$$P \in M \leftrightarrow (EX)[X \in M' \;\&\; (k)\{p_k = M_k(x_1, x_2, \ldots, x_k)\}].$$

The following notation has been applied:

$x_1, x_2, \ldots, x_k, \ldots$ are natural numbers;

X is the sequence $x_1, x_2, \ldots, x_k, \ldots$;

$p_1, p_2, \ldots, p_k, \ldots$ are *"figures"*;
P is the sequence $p_1, p_2, \ldots, p_k, \ldots$;
$\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots$ are sets of natural numbers;
$\tau\alpha$ is an element, arbitrarily chosen from α;
$M'_{k+1}(x_1, x_2, \ldots, x_k)$ is a function the values of which are sets of natural numbers;
$M_k(x_1, x_2, \ldots, x_k)$ is a function the values of which are "figures".

So every spread is determined by two sequences M' and M of functions. Therefore, the question arises as to how these sequences should be defined. It seems obvious that they should be defined in terms of recursive procedures. I shall not go into this matter, which has been investigated by Kleene (1950 and later).

145. Semantic Construction of Intuitionistic Logic

The considerations in the preceding Section go back to 1945; they were published in 1947. Only recently, however, I have been able to substantiate the claim which I made concerning the application of the semantic method in an analysis of intuitionistic logic and mathematics. In this Section, I wish to sum up my results on intuitionistic elementary logic.

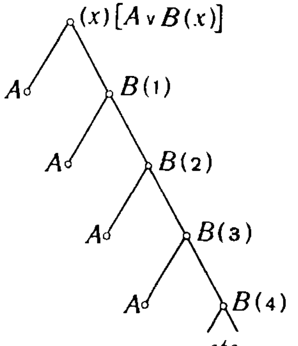

Fig. 11

(1) Let us first consider a concrete example. The tree in fig. 11 clearly realises, in a sense to be analysed later, the sentence:

(a) $$(x)[A \vee B(x)],$$

both from a classical and from an intuitionistic point of view. Is it correct to say that, in the same sense, it also realises the sentence:

(b) $$A \vee (x)B(x)?$$

CLASSICALLY, we are inclined to answer in the affirmative. For each branch in our tree represents a certain model of sentence (a), and there are clearly two kinds of branches. The (infinitely many) branches of the first kind represent models of the sentence A. In addition, there is one branch of a different kind, which represents a model of the sentence $(x)B(x)$. Therefore, every branch in our tree represents a certain model of the sentence (b), in accordance with the fact that the sequent:

$$(x)[A \vee B(x)] \vdash A \vee (x)B(x)$$

is classically valid.

INTUITIONISTICALLY, on the other hand, we would argue as follows. Suppose a subject is given the instruction to make, for $k = 1, 2, \ldots$, a free choice between A and $B(k)$. Once the subject has chosen A, we lose interest in any further choices. It will be clear that each choice sequence determines some branch in our tree and thus realises the sentence $(x)[A \vee B(x)]$. Nevertheless, we may forever remain uncertain as to the decision between A and $(x)B(x)$. For, if our subject happens to choose $B(1)$, $B(2)$, ..., without committing himself to continue in this manner, then neither A nor $(x)B(x)$ is ever "*secured*", and so it makes no sense to say that $A \vee (x)B(x)$ is realised.

Now, as observed in Section 139, the subjective element which is brought into the situation by the introduction of choice sequences can be eliminated if we agree to concentrate upon such properties of choice sequences as appear after a finite number of choices. This attitude implies, however, a radical change in the semantical notions. The classical rules determine (in our present terminology) the validity or non-validity of a formula *on each branch separately*; for this reason they entail the above difficulties, which are at the bottom of the divergences between classical and intuitionistic views. These difficulties vanish, if we agree to determine validity or non-validity, not on individual branches, but collectively on all those branches which have a certain initial segment in common, that is, *on a subtree.*

Therefore, we shall study trees of a special kind, of which the above one provides an example. With certain points of such a tree, we connect a formula or, more generally, a *junctive*. – For each finite sequence of formulas $X_1, X_2, \ldots, X_k$, we introduce (following R. Carnap, 1943) a *conjunctive* $[X_1, X_2, \ldots, X_k]$ and a *disjunctive* $\{X_1, X_2, \ldots, X_k\}$. We shall not always distinguish $[X]$ and $\{X\}$ from each other or from X, but we must always distinguish $[\emptyset]$ from $\{\emptyset\}$, where $\emptyset$ is the empty sequence. –

(2) We denote as a *tree*, every sextuple $\mathbf{M} = <S, O, P, R, f, F>$, formed by a set S (the elements $p, q, \ldots$ of which are called *points*), two special elements O and P of S (called, respectively, the *origin* and the *vertex* of $\mathbf{M}$), a relation R whose field is S, a function f which with each point p on $\mathbf{M}$ associates a natural number $f(p)$ as its *rank*, and a function F which maps some subset S' of S on some other set, such that:

(i) for every point p the set of all points q with $R(p, q)$ is finite;
(ij) for every point $q \neq O$, there is exactly one point p with $R(p, q)$;
(iij) $f(O) = 1$; for $1 \leqq k \leqq f(P)$, there is exactly one point p with $f(p) = k$;
(iv) if $R(p, q)$, then $f(q) = f(p) + 1$.

In addition, we admit a *zero-tree* which does not contain any point. – If there is a largest number k which appears as rank of a point, then k is called the *length* of $\mathbf{M}$, and $\mathbf{M}$ is said to be *of finite length.*

We denote as a *branch* of the tree $\mathbf{M}$, any maximal sequence of points $p_1, p_2, \ldots, p_{k-1}, p_k, \ldots$ such that, for every subscript k, we have $R(p_{k-1}, p_k)$. – If there is a largest subscript k, then k is called the *length* of the branch.

If p is a point on $\mathbf{M}$, then the *subtree* $\mathbf{M}^{(p)}$ will be the tree $<S', O, p, R', f', F'>$, where S' is the set of all points q contained in some branch of $\mathbf{M}$ which contains p, and where R', f', and F' are, respectively, the restrictions of R, f, and F to S'.

We shall say that $\mathbf{M}$ is the *union* of its (finitely many) subtrees $\mathbf{M}', \mathbf{M}'', \ldots$ (or that $\mathbf{M}$ is *decomposed into* these subtrees), if S is the union of the sets $S', S'', \ldots$.

The *trunk* $\mathbf{M}_k$ of length k of $\mathbf{M}$ is the tree $<S_k, O, P', R', f', F'>$ where S_k is the set of all points p of rank $f(p) \leqq k$, where $P' = P$ if $f(P) \leqq k$ and $P' = O$ otherwise, and where R', f', and F' are the restrictions of R, f, and F to S_k. As a *construction* of a tree $\mathbf{M}$, we denote every sequence of trunks $\mathbf{M}_k$ of $\mathbf{M}$, the lengths k of which have an upper bound k_0 if and only if k_0 is the length of $\mathbf{M}$.

We shall use a number of elementary theorems on trees which are, however, so obvious that it is not necessary to prove or even to state them.

(3) A *semi-model* **M** is a tree such that the values $F(p)$ of the function F are (formulas or) junctives. – We shall often say that the (formula or) junctive $F(p)$ is *connected* with the point p or *appears* on the semi-model **M**.

A formula X is said to be *valid* on the semi-model **M** (and **M** is said to *fulfil* X), whenever one of the following conditions is satisfied:
(i) X is an atom and **M** is the union of finitely many subtrees $\mathbf{M}^{(p)}, \mathbf{M}^{(p')}, \mathbf{M}^{(p'')}, \ldots$, such that with each of the vertices $p, p', p'', \ldots$ either X itself or some conjunctive in which X occurs is connected;
(ij) X is $\overline{Y}$ and Y is not valid on any subtree $\mathbf{M}'$ of **M**;
(iij) X is $Y \& Z$ and both Y and Z are valid on **M**;
(iv) X is $Y \vee Z$ and **M** is the union of finitely many subtrees on each of which either Y or Z is valid;
(v) X is $Y \to Z$ and, whenever Y is valid on a subtree $\mathbf{M}'$ of **M**, Z is also valid on $\mathbf{M}'$;
(vi) X is $(x)Y(x)$ and each of the formulas $Y(1)$, $Y(2)$, ... is valid on **M**;
(vij) X is $(Ex)Y(x)$ and **M** is the union of finitely many subtrees on each of which some formula $Y(k)$ is valid;
(viij) X is any formula and **M** is the zero-tree.

In accordance with this definition, the tree **M** in fig. 11 is a semi-model, and it fulfils the formula $(x)[A \vee B(x)]$. For, as shown in

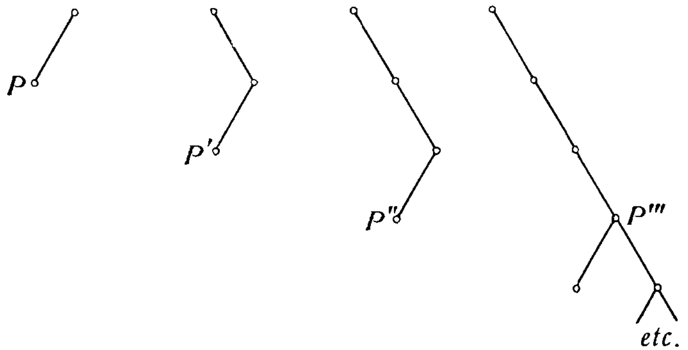

FIG. 12

fig. 12, $\mathbf{M}$ can be decomposed into four subtrees $\mathbf{M}^{(p)}, \mathbf{M}^{(p')}, \mathbf{M}^{(p'')}$, and $\mathbf{M}^{(p''')}$, such that, by (i), $A[=F(p)=F(p')=F(p'')]$ is valid on $\mathbf{M}^{(p)}, \mathbf{M}^{(p')}$, and $\mathbf{M}^{(p'')}$, whereas, again by (i), $B(3)$ $[=F(p''')]$ is valid on $\mathbf{M}^{(p''')}$. So, by (iv), $A \vee B(3)$ is valid on $\mathbf{M}$. As, likewise, $A \vee B(1)$, $A \vee B(2)$, $A \vee B(4)$, ... are all valid on $\mathbf{M}$, $(x)[A \vee B(x)]$ is valid on $\mathbf{M}$ by (vi). On the other hand, it is easy to show that $A \vee (x)B(x)$ is not valid on $\mathbf{M}$. –

A conjunctive $[X_1, X_2, \ldots, X_k]$ is said to be *valid* on a semi-model $\mathbf{M}$, whenever each of the formulas $X_1, X_2, \ldots, X_k$ is valid on $\mathbf{M}$; and a disjunctive $\{X_1, X_2, \ldots, X_k\}$ is said to be *valid* on $\mathbf{M}$, whenever $\mathbf{M}$ is the union of finitely many subtrees, on each of which one of the formulas $X_1, X_2, \ldots, X_k$ is valid.

Theorem. If a formula or a junctive is valid on a semi-model $\mathbf{M}$, then it is valid on every subtree $\mathbf{M}'$ of $\mathbf{M}$.

Theorem. If a semi-model $\mathbf{M}$ is decomposed into a finite number of subtrees $\mathbf{M}', \mathbf{M}''$, ..., on all of which a certain formula or junctive is valid, then that formula or junctive is also valid on $\mathbf{M}$ itself.

These theorems are easily proved by recursion on the construction of a formula or junctive, using the fact that the intersection of two subtrees of a given tree is either the zero-tree (if there is no branch which contains the two vertices of the subtrees) or one of these subtrees (namely, the one whose vertex is of higher rank).

A semi-model $\mathbf{M}$ is called a *model* if it is not the zero-tree and if, whenever a junctive or a formula is connected with a point p on $\mathbf{M}$, that junctive or formula is valid on the subtree $\mathbf{M}^{(p)}$ of $\mathbf{M}$.

Note that a formula X, if it is not an atom, may be valid on a semi-model $\mathbf{M}$, even though it does not appear on $\mathbf{M}$. – The semi-model $\mathbf{M}$ in fig. 11 clearly is a model.

(4) Let us agree to say that the sequent:

$$\text{(f)} \qquad U_1, U_2, \ldots, U_m \vdash V_1, V_2, \ldots, V_n$$

holds true (intuitionistically) if, whenever a model $\mathbf{M}$ fulfils the conjunctive $[U_1, U_2, \ldots, U_m]$, it also fulfils the disjunctive $\{V_1, V_2, \ldots, V_n\}$; in other words: if every model $\mathbf{M}$ which fulfils all formulas $U_1, U_2, \ldots, U_m$ can be decomposed into finitely many submodels each of which fulfils at least one of the formulas $V_1, V_2, \ldots, V_n$.

A model **M** is said to provide a *counter-model* to the sequent (f), if it fulfils the conjunctive $[U_1, U_2, \ldots, U_m]$ even though it does not fulfil the disjunctive $\{V_1, V_2, \ldots, V_n\}$.

The model in fig. 11 clearly provides a counter-model to the sequent:

$$(x)[A \vee B(x)] \vdash A \vee (x)B(x).$$

Essentially the same model was constructed by Mostowski (1948). – We now consider the following set of rules.

$$\text{(i)}\quad K', Z, K'' \vdash Z$$

$$\text{(ij}^a\text{)}\quad \frac{K, \overline{Y} \vdash L, Y}{\overline{Y}, K \vdash L} \qquad \text{(ij}^b\text{)}\quad \frac{K, Y \vdash \emptyset}{K \vdash \overline{Y}}$$

$$\text{(iij}^a\text{)}\quad \frac{K, Y, Z \vdash L}{Y \,\&\, Z, K \vdash L} \qquad \text{(iij}^b\text{)}\quad \frac{K \vdash L, Y \textit{ et } K \vdash L, Z}{K \vdash Y \,\&\, Z, L}$$

$$\text{(iv}^a\text{)}\quad \frac{K, Y \vdash L \textit{ et } K, Z \vdash L}{Y \vee Z, K \vdash L} \qquad \text{(iv}^b\text{)}\quad \frac{K \vdash L, Y, Z}{K \vdash Y \vee Z, L}$$

$$\text{(v}^a\text{)}\quad \frac{K, Y \to Z \vdash L, Y \textit{ et } K, Z, Y \to Z \vdash L}{Y \to Z, K \vdash L} \qquad \text{(v}^b\text{)}\quad \frac{K, Y \vdash Z}{K \vdash Y \to Z}$$

$$\text{(vi}^a\text{)}\quad \frac{K, Y(1), \ldots, Y(p), (x)Y(x) \vdash L}{(x)Y(x), K \vdash L} \qquad \text{(vi}^b\text{)}\quad \frac{K \vdash Y(p)}{K \vdash (x)Y(x)}$$

$$\text{(vij}^a\text{)}\quad \frac{K, Y(p) \vdash L}{(Ex)Y(x), K \vdash L} \qquad \text{(vij}^b\text{)}\quad \frac{K \vdash L, Y(1), \ldots, Y(p), (Ex)Y(x)}{K \vdash (Ex)Y(x), L}$$

$$\text{(viij)}\quad \frac{K \vdash Z_1 \textit{ vel } K \vdash Z_2, \ldots, Z_k, Z_1}{K \vdash Z_1, Z_2, \ldots, Z_k}$$

These rules may be considered under various aspects.

(I) If read upside down, they show how in successive steps to convert a given model of a certain conjunction into a model of a certain disjunction. This point will be discussed under (6).

(II) As they stand, they rather suggest a certain Calculus of Sequents related to the Formal System F which we studied in Section 92; this point will be taken up under (5).

(III) If read upside down, the above rules may also be construed as instructions for the construction of a semantic tableau. This point I wish to discuss at once.

If we are interested in a certain sequent (f), then we start the construction of a semantic tableau for (f) by inserting $U_1, U_2, \ldots, U_m$ as initial formulas in the left column and $V_1, V_2, \ldots, V_n$ as initial formulas in the right column. The development of the tableau is carried out under the above rules (i)–(viij); the effect of an application of each rule can be described as follows.

ad (i) This is the rule for the closure of a tableau.

ad (ij^b) Rule (ij^a) is as in the classical case, but with respect to rule (ij^b) there are two differences. In the first place, this rule can only be applied, if the right column contains no other formulas besides $\overline{Y}$; in the second place, after Y has been inserted in the left column, the formula $\overline{Y}$ in the right column may be cancelled. – After an application of rule (ij^a), however, the formula $\overline{Y}$ in the left column is *not* to be cancelled. This is not meant to make things more difficult; on the contrary, we shall see that we sometimes must submit a given formula to several applications of the same rule in order to bring about the closure of a tableau or subtableau.

ad (iij^b), (iv^a), (v^a) As a result of an application of these rules, the tableau splits up into two subtableaux, the closure of both of which is required for the closure of the original tableau. For this reason we shall say that the resulting subtableaux are *conjunctively connected.*

ad (vi^b) and (vij^a) As in the classical case, each application of these rules demands the introduction of a "*fresh*" numeral.

ad (vi^a) and (vij^b) In these cases, all numerals hitherto introduced must be used; the formulas $(x)Y(x)$ and $(Ex)Y(x)$, respectively, are retained for further applications of the same rule.

ad (viij) In this case, the tableau splits up into two *disjunctively connected* subtableaux, the closure of one of which is sufficient to bring about the closure of the original tableau.

As an example, we construct the semantic tableau for the sequent:

$$\emptyset \vdash A \vee \bar{A}.$$

Valid	(*Not valid?*)
	$A \vee \bar{A}$
	$A, \bar{A}$ (iv^b)
	A $\quad \bar{A}, A$ (viij)
	$\bar{A}$ $\vdots$ $A, \bar{A}$ (viij)
(ij^b) A	$\emptyset$ $\vdots$ A $\vdots$ $\bar{A}, A$ (viij)
	$\bar{A}$ $\vdots$ $A, \bar{A}$ (viij)
(ij^b) A	$\emptyset$ $\vdots$ ...

It needs hardly saying that this tableau will never be closed; this is in accordance with the fact that the sequent under consideration does not hold true intuitionistically. – Let us also construct the semantic tableau for the sequent:

$$\emptyset \vdash \overline{\overline{A \vee \bar{A}}}.$$

Valid	(*Not valid?*)
	$\overline{\overline{A \vee \bar{A}}}$
(ij^b) $\overline{A \vee \bar{A}}$	$\emptyset$
	$A \vee \bar{A}$ (ij^a)
	$A, \bar{A}$ (iv^b)
	A $\quad \bar{A}, A$ (viij)
	$\bar{A}$ $\vdots$ $A, \bar{A}$ (viij)
(ij^b) A	$\emptyset$
	$A \vee \bar{A}$ (ij^a)
	$A, \bar{A}$ (iv^b)
	A $\vdots$ $\bar{A}, A$ (viij)

In this case the tableau is closed, thanks to the possibility of applying rule (ij^a) twice. The sequent under consideration indeed holds true intuitionistically.

(5) As in the classical case, a closed semantic tableau for a sequent (f) may be said to constitute a derivation of this sequent in

a certain Formal System F_0. Again it is possible to give this system the shape of a regular System N or L of Gentzen type, and to show the equivalence of the various systems thus obtained with other extant formalisations of intuitionistic logic. In view of the explanations contained in Section 92, it will not be necessary to go once more into a detailed discussion of this matter.

(6) It is, however, highly significant that it is also possible to construe a semantic tableau as description of a tentative construction of a mathematical character, the closure of the tableau announcing the successful completion of the construction.

In order to prove that a certain sequent (f) holds true intuitionistically, we have to point out, essentially, that there is a certain method which enables us effectively to carry out the decomposition of an arbitrary model **M** of all formulas $U_1, U_2, \ldots, U_m$ into finitely many submodels each of which fulfils one of the formulas $V_1, V_2, \ldots, V_n$. In other words, we have to convert the given model **M** of $[U_1, U_2, \ldots, U_m]$ into a model N of $\{V_1, V_2, \ldots, V_n\}$. Each of the rules (i)–(viij) may be taken to represent a certain step in the construction of N.

ad (i) In this case, the model N of $\{Z\}$ may clearly be identified with the given model **M** of $[K', Z, K'']$.

ad (ij^a) Suppose we know how to convert a given model $\mathbf{M}_1$ of $[K, \overline{Y}]$ into a model N_1 of $\{L, Y\}$; let us take **M** as $\mathbf{M}_1$ and let us consider the resulting model N_1. As N_1 fulfils $\{L, Y\}$, it can be decomposed into two parts N_2 and N_3 which fulfil, respectively, $\{L\}$ and $\{Y\}$. But N_3 must also fulfil $\overline{Y}$ and so must be the zero-tree. It follows that N_2 can be taken as N. – The discussion of rule (ij^b) follows similar lines.

ad (v^a) Suppose we know how to convert a model $\mathbf{M}_1$ of $[K]$ into a model N_1 of $\{L, Y\}$ and to convert a model $\mathbf{M}_2$ of $[K, Z]$ into a model N_2 of $\{L\}$; let **M** be a model of $[Y \rightarrow Z, K]$. We take **M** as $\mathbf{M}_1$ and we decompose the resulting model N_1 into submodels N′ and N″ of $\{L\}$ and Y, respectively. As N″, being part of **M**, fulfils $Y \rightarrow Z$, it fulfils Z. We now take N″ as $\mathbf{M}_2$. Then clearly **M** is converted into the union of N′ and N_2, which both fulfil $\{L\}$. – The remaining cases are treated likewise.

ad (viij) If a model **M** of $[K]$ is to be converted into a model N of $\{Z_1, Z_2, \ldots, Z_k\}$, then clearly two methods are available. In the first place, we may try to convert **M** into a model N_1 of $\{Z_1\}$ but we may also try to decompose **M** into finitely many subtrees **M**′, **M**″, ... which can be converted, respectively, into subtrees **N**′, **N**″, ... of $\mathbf{N}_2$ such that **N**′, **N**″, ... are models of $Z_2, Z_3, \ldots, Z_k$, or Z_1.

As an illustration, let us first consider the above construction for the sequent: (see fig. 13).

$$\varnothing \vdash \overline{\overline{\overline{A \vee \bar{A}}}}.$$

(a) We wish to convert every model **M** of $[\varnothing]$, that is, every model **M** whatsoever, into a model **N** of $\overline{\overline{\overline{A \vee \bar{A}}}}$; (b) it suffices to convert every subtree **M**′ of **M** which fulfils $\overline{A \vee \bar{A}}$ into a model of $\{\varnothing\}$, that is, into the zero-tree; (c) this construction can be carried out by converting **M**′ into a model of $A \vee \bar{A}$, or $\{A, \bar{A}\}$; (d) thus we try to convert **M**′ either into a model of A, or into a model of $\{\bar{A}, A\}$; (e) the first approach does not meet with immediate success; (f) the second

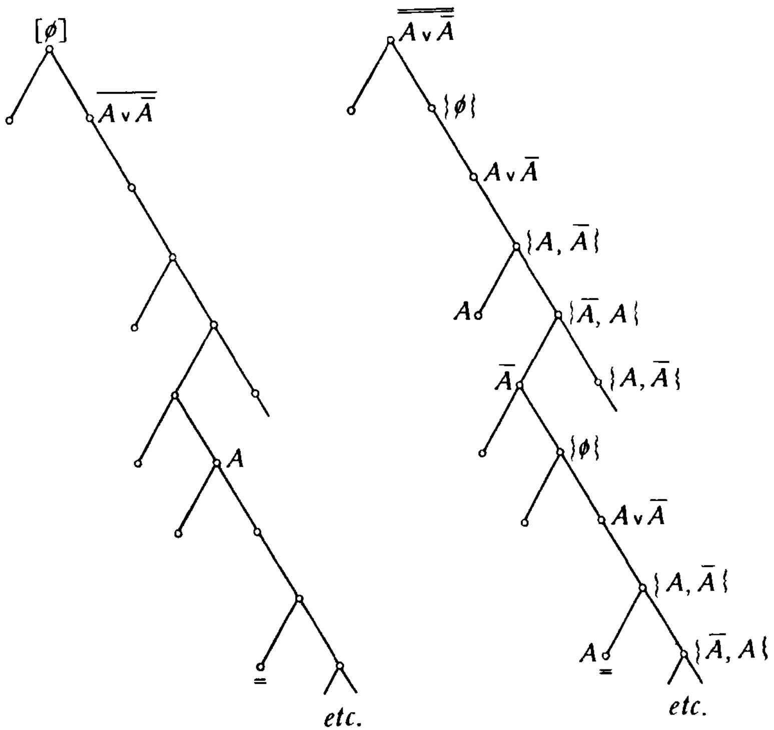

FIG. 13

approach demands that we convert $\mathbf{M}'$ into a model of $\bar{A}$ or into a model of $\{A, \bar{A}\}$; (g) to convert $\mathbf{M}'$ into a model of $\bar{A}$, it is sufficient to convert every subtree $\mathbf{M}''$ of $\mathbf{M}'$ which fulfils A into the zero-tree; (h) as $\overline{A \vee \bar{A}}$ is valid on $\mathbf{M}'$ and hence on its subtree $\mathbf{M}''$, we resort once again to steps (c) and following; and now step (e) is found to be successful. –

Our second example is concerned with the sequent:

$$\varnothing \vdash A \vee \bar{A}.$$

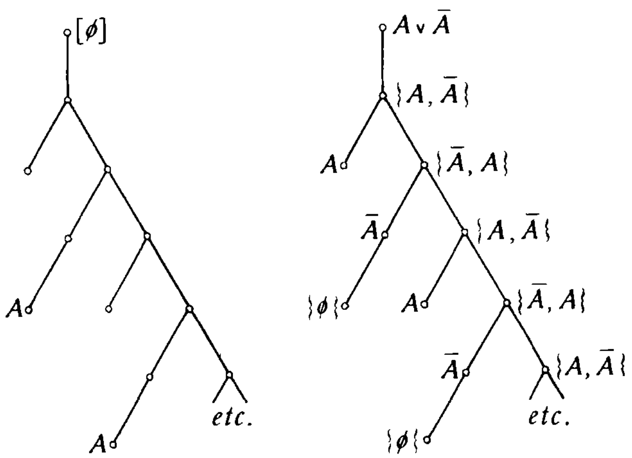

FIG. 14

In this case, the construction, as shown in the figure, is not successful. However, the figure shows still more: the tree $\mathbf{M}$ which results from the construction is clearly a counter-example to the sequent under consideration.

One might anticipate a similar situation in all such cases; that is, one might expect the following completeness theorem for our Formal System $\mathrm{F_0}$: whenever a sequent is not derivable in $\mathrm{F_0}$, then there is a counter-model to it. However, such a completeness theorem is not intuitionistically provable; but we are able to prove a statement which is classically equivalent to it, though intuitionistically weaker.

(7) We suppose that the construction of semantic tableaux has been suitably normalised; this can be achieved by adding the rule:

(ix) $$\frac{K,\ Y \vdash L}{Y,\ K \vdash L} \quad (\textit{if } Y \textit{ is an atomic formula})$$

and fixing a certain alternating order for the successive applications of rules (ij)–(ix). Then the semantic tableau for a given sequent (f), or $C \vdash D$, is uniquely determined. As in Section 70, *sub* (1), and in Section 92, *sub* (6), we consider nested strings T' of subtableaux of T, with the understanding, however, that we make a choice only between conjunctively connected subtableaux; if a tableau splits up into two disjunctively connected subtableaux, then both subtableaux are included in T'.

As shown in the above example, the tableau T can be represented by an ordered couple $<\mathbf{M}, \mathbf{N}>$ of semi-models; the ordered couple $<\mathbf{M}, \mathbf{N}>$ will be called the *Herbrand field* for the sequent $C \vdash D$ under consideration. To each string T' in T there obviously corresponds a certain ordered couple $<\mathbf{M}', \mathbf{N}'>$ of semi-models. $\mathbf{M}'$ is part of $\mathbf{M}$ but not necessarily a subtree of $\mathbf{M}$; likewise for $\mathbf{N}'$ and $\mathbf{N}$. We take $\mathbf{M} = <S, O, O', R, f, F>$, $\mathbf{N} = <S, O, O', R, f, G>$, $\mathbf{M}' = <S', O, O'$ $R, f, F>$, and $\mathbf{N}' = <S', O, O', R, f, G>$.

(8) As an introduction to our completeness proof, we shall first prove the following

Lemma: *Let* $\mathbf{M}$, $\mathbf{N}$, $\mathbf{M}'$, *and* $\mathbf{N}'$ *be as above and suppose the corresponding string* T' *in* T *not to contain a closure; let* p *be any point in* S'*; then:*

(A) $F(p)$ *is valid on* $\mathbf{M}'^{(p)}$*; and*

(B) $G(p)$ *cannot be valid on* $\mathbf{M}'^{(p)}$.

Consequently, $\mathbf{M}'$ *is a model and, specifically,* $\mathbf{M}'$ *is a counter-model to the sequent* (f). –

Proof. We first observe that (A) is trivial if $F(p)$ is $[\emptyset]$ and that (B) is trivial if $G(p)$ is $\{\emptyset\}$.

Moreover, by the construction of T, if an atomic formula X occurs in $F(p)$, then it also occurs in $F(q)$ for every point q on $\mathbf{M}'^{(p)}$ with $f(q) > f(p)$; and if an atomic formula X occurs in $G(p)$, then we can find a point q on $\mathbf{M}'^{(p)}$ such that $f(q) > f(p)$ and $G(q) = X$.

We consider the following statements:

(A′) *If* X *occurs in* $F(p)$, *then* X *is valid on* $\mathbf{M}'^{(p)}$;

(B′) *If* X *occurs in* $G(p)$, *then* X *cannot be valid on every subtree* $\mathbf{M}^0$ *of* $\mathbf{M}'^{(p)}$;

(A″) *If X is the first formula in $F(p)$, then X is valid on* $\mathbf{M}'^{(p)}$;
(B″) *If $G(p)$ is X, then X cannot be valid on every subtree* $\mathbf{M}^0$ *of* $\mathbf{M}'^{(p)}$.

In the first place, we prove (A″) and (B″) simultanuously by recursion on the construction of X. We only consider those particular cases which present a certain specific interest; the remaining cases are simpler, but they are treated in essentially the same manner; one may also compare the completely similar discussion under (4).

ad (i^a) If the first formula X in $F(p)$ is an atomic formula, then it is clearly valid on $\mathbf{M}'^{(p)}$.

ad (i^b) Let $G(p)$ be an atomic formula X; now if X were valid on every subtree $\mathbf{M}^0$ of $\mathbf{M}'^{(p)}$, then clearly some closure would arise in T'; this contradicts the hypothesis of our Lemma, so X cannot be valid on every subtree $\mathbf{M}^0$ of $\mathbf{M}'^{(p)}$.

ad (ij^a) Suppose that the first formula X in $F(p)$ is $\overline{Y}$, whereas the statements (A″) and (B″) hold true in regard to Y. In fig. 15,

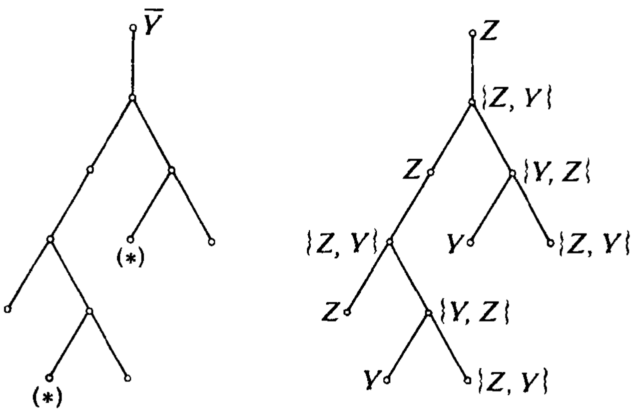

FIG. 15

we give a rough sketch of the structure of $\mathbf{M}'^{(p)}$ and $\mathrm{N}'^{(p)}$. By our supposition, Y cannot be valid on the subtrees marked (*) and, by the construction of $\mathbf{M}'$ and N', such subtrees must arise for each subtree $\mathbf{M}^0$ of $\mathbf{M}'^{(p)}$; hence $\overline{Y}$ is valid on $\mathbf{M}'^{(p)}$.

ad (iv^b) Suppose that $G(p)$, or X, is $Y \vee Z$, whereas in regard to Y and Z the statements (A″) and (B″) hold true. In fig. 16, a rough sketch is given of the structure of $\mathbf{M}'^{(p)}$ and $\mathrm{N}'^{(p)}$; the same structure belongs, of course, to any relevant subtrees $\mathbf{M}^0$ and $\mathbf{N}^0$. It will be clear that $Y \vee Z$ cannot be valid on every subtree $\mathbf{M}^0$ of $\mathbf{M}'^{(p)}$.

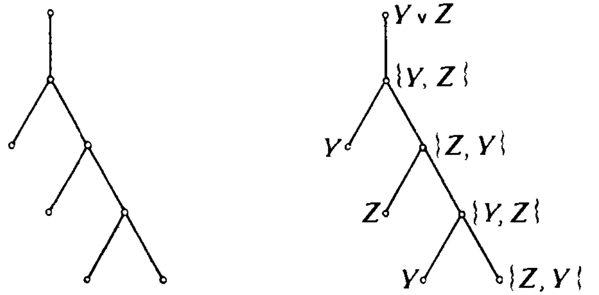

FIG. 16

ad (vij^b) Suppose that $G(p)$, or X, is $(Ex)Y(x)$, whereas in regard to $Y(1)$, $Y(2)$, ... the statements (A″) and (B″) hold true. In fig. 17, we give a rough sketch of the structure of subtrees $\mathbf{M}^0$ and $\mathbf{N}^0$ of $\mathbf{M}'^{(p)}$ and $\mathbf{N}'^{(p)}$, respectively. It follows clearly that $(Ex)Y(x)$ cannot be valid on every subtree $\mathbf{M}^0$ of $\mathbf{M}'^{(p)}$. –

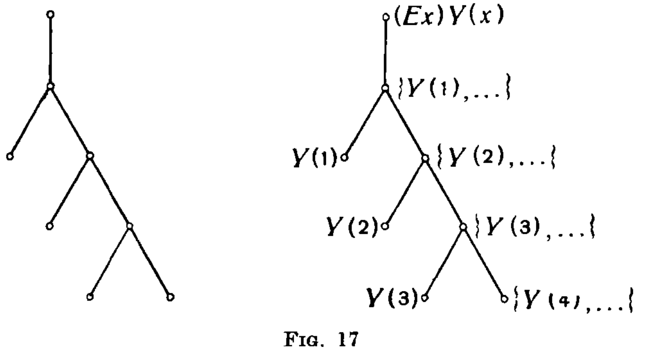

FIG. 17

We may conclude that the statements (A″) and (B″) hold true for every formula X. It is easy to see that, hence, the statements (A′) and (B′) also hold true for every formula X; from this it follows that also the statements (A) and (B) hold true. This completes the proof of our Lemma; *cf.* the two Theorems on p. 448.

(9) It will be convenient to adopt the following terminology. The above semi-models $\mathbf{M}'$ will be called *approximate counter-models* to the sequent (f). An approximate counter-model $\mathbf{M}'$ is said to be *of depth* k, if and only if the upper part T'_k of T' which corresponds to the trunk $\mathbf{M}'_k$ of $\mathbf{M}'$ contains a closure, whereas $\mathbf{M}'_{k-1}$ does not. If no such k can be found, then $\mathbf{M}'$ is said to be *of infinite depth.*

Then our above Lemma can be stated more briefly as follows:

If an approximate counter-model $\mathbf{M}'$ *to a sequent* (f) *is of infinite depth, then* $\mathbf{M}'$ *is a counter-model to* (f).

On the other hand, let an approximate counter-model $\mathbf{M}'$ to the sequent:

(f) $$C \vdash D$$

be of depth k. Let $\mathbf{M}^*$ be any model such that $\mathbf{M}^*_k$ is exactly like $\mathbf{M}'_k$ with the understanding, however, that those formulas (or junctives) appearing on $\mathbf{M}'_k$, which have arisen from D, do not appear on $\mathbf{M}^*_k$. It will be clear that:

The model $\mathbf{M}^*$ *fulfils the conjunctive* C.

(10) We can now state the following classical version of our completeness theorem for the Formal System F_0:

For every sequent (f), *exactly one of the following conditions is satisfied*:

(i) *The semantic tableau for* (f) *is closed, and hence* (f) *holds true* (*intuitionistically*) *and is derivable in* F_0;

(ij) *The semantic tableau for* (f) *is not closed; hence there is a counter-model* $\mathbf{M}'$ *to* (f), *and* (f) *is not derivable in* F_0.

Proof. It is been shown under (4)–(6) that, whenever the semantic tableau for a sequent (f) is closed, (f) holds true and is derivable in F_0. The arguments used were also conclusive from an intuitionistic point of view. –

Now let us consider all approximate counter-models $\mathbf{M}'$ to (f). There is either an upper bound b for all depths k of approximate counter-models $\mathbf{M}'$ to (f), or their is no such upper bound.

(i) If there is an upper bound b as above, then clearly the upper part T'_b of each string T', in the semantic tableau T for (f) must contain a closure. It follows that the semantic tableau T is closed.

(ij) If there is no such upper bound then, by the tree theorem (*cf.* Section 69), there must be an approximate counter-model $\mathbf{M}'$ of infinite depth. By our above Lemma, $\mathbf{M}'$ is a counter-model to (f).

(11) From an intuititionistic point of view, the above argument involves two objectionable steps. In the first place, we have started by applying the principle of the excluded third; secondly, the tree theorem is not available in intuitionistic mathematics.

We shall meet the first objection by giving a statement of the completeness theorem which is intuitionistically weaker; this weaker statement can then be proved by an argument which is a variant to Brouwer's proof of his Fundamental Theorem on Bounded Spreads (*cf.* Section 140).

If a sequent:

(f) $$C \vdash D$$

holds true intuitionistically, then it is derivable in the Formal System F_0.

Proof. Suppose that the sequent (f) holds true. This means, intuitionistically, that we have an effective procedure which, whenever a model $\mathbf{M}$ of C is given, converts it into a model $\mathbf{N}$ of D. The elementary steps in such a procedure were summed up under (6).

(As these steps correspond to the steps in a formal derivation, one might be inclined to consider our argument as trivial, or even circular. However, such a reaction would not be justified. The given procedure applies to models $\mathbf{M}$ individually, whereas a formal derivation provides, so to speak, one uniform treatment for all models $\mathbf{M}$ collectively. Our argument shows, essentially, that all individual applications of the given procedure can be merged into one uniform treatment.)

Actually, the procedure does not operate on a model $\mathbf{M}$ itself, but rather on a certain trunk $\mathbf{M}_k$ of it; a trunk $\mathbf{M}_k$ of suitable length is converted into a trunk N_k of N. Therefore, the procedure also applies to trunks $\mathbf{M}_k$ which belong approximately, though not actually, to models $\mathbf{M}$ of C. If the procedure fails to work, it is either because $\mathbf{M}_k$ is of insufficient length or because $\mathbf{M}_k$ cannot be extended into a model $\mathbf{M}$ of C.

(This does not imply the existence of a decision procedure. It may happen that $\mathbf{M}_k$ is converted into N_k, even though it cannot be extended into a model $\mathbf{M}$ of C.)

As the procedure may be tentatively applied to *every* trunk $\mathbf{M}_k$, we see that there is a function b of trunks to natural numbers, such that:

$b(\mathbf{M}_k) = 1$, if the procedure transforms $\mathbf{M}_k$ into $\mathbf{N}_k$;

$b(\mathbf{M}_k) = 3$, if the procedure fails because $\mathbf{M}_k$ cannot be extended into a model $\mathbf{M}$ of C;

$b(\mathbf{M}_k) = 5$, if the procedure fails on account of the fact that $\mathbf{M}_k$ is not of sufficient length.

From this point on, we essentially duplicate Brouwer's above-mentioned proof. –

We shall not consider all possible trunks $\mathbf{M}_k$, but only those which belong to approximate counter-models $\mathbf{M}'$. We observe that the values $b(\mathbf{M}'_k)$ of the function b must be (i) *appropriate*, (ij) *justifiable*, and (iij) *effectively computable* for every argument value $\mathbf{M}'_k$.

ad (i) if $b(\mathbf{M}'_k) = 1$, then the transformation of $\mathbf{M}'_k$ into $\mathbf{N}'_k$ must be effectively available; if $b(\mathbf{M}'_k) = 3$, then we must be able to find out what is wrong with $\mathbf{M}'$; and if $b(\mathbf{M}'_k) = 5$, then by sufficiently extending $\mathbf{M}'_k$ we must always finally arrive at some $\mathbf{M}'_j$ such that $b(\mathbf{M}'_j) = 1$ or 3.

ad (ij) We now introduce a second function, c, as follows. If $b(\mathbf{M}'_k) = 1$ and if the decomposition of $\mathbf{M}'$ can be seen from $\mathbf{M}'_k$, then we take $c(\mathbf{M}'_k) = 1$. But if this is not the case, then we take $c(\mathbf{M}'_k) = 2$.

If $b(\mathbf{M}'_k) = 3$, and if $\mathbf{M}'_k$ contains two points p and q such that q is on $\mathbf{M}'^{(p)}$ and that some formula X occurs both in $F(p)$ and in $G(q)$, then we take $c(\mathbf{M}'_k) = 3$. But if this is not the case, then we take $c(\mathbf{M}'_k) = 4$.

If $b(\mathbf{M}'_k) = 5$, then we take $c(\mathbf{M}'_k) = 5$.

ad (iij) Clearly the value of b or c can only be directly found if $b(\mathbf{M}'_k) = c(\mathbf{M}'_k) = 1$ or 3. It all other cases, its computation will involve reference to certain other values of these functions.

Now we consider the trunk $\mathbf{M}'_1$ which all approximate counter-models $\mathbf{M}'$ have in common, and we compute $b(\mathbf{M}'_1)$. Let r_0 be the largest of all (*finitely many*) numbers k, such that some value $b(\mathbf{M}'_k)$ or $c(\mathbf{M}'_k)$ is referred to in the computation of $b(\mathbf{M}'_1)$. It is easy to see that r_0 is an upper bound for the depth k of an approximate counter-model $\mathbf{M}'$. For suppose $\mathbf{M}'$ to be of depth $k > r_0$. Then clearly $c(\mathbf{M}'_{r_0}) = 2$, 4, or 5; it would follow that our computation of $b(\mathbf{M}'_1)$ was not justified.

As r_0 is an upper bound for the depth k of an approximate counter-model $\mathbf{M}'$ to (f), it follows by the argument in (10), under (i) that the semantic tableau T for (f) must be closed. – This completes our proof.

(12) The above methods lead to simplified proofs of known results, but they may also provide a starting-point for further investigations.

Although it is not possible here to go more deeply into these matters, I wish to make a few remarks in connection with certain objections to the intuitionistic version of the above completeness theorem; it will not be necessary here to state these objections which have been raised, from different viewpoints, by A. Heyting and by K. Gödel and G. Kreisel.

In my opinion, the difficulties are connected rather with the statement of the completeness theorem than with its proof. The hypothesis in the theorem can be restated as follows:

All models $\mathbf{M}'$ *which fulfil the conjunctive* C *also fulfil the disjunctive* D.

However, we have not established a construction which yields the totality of all these models $\mathbf{M}'$ and to which the above hypothesis can thus be construed to refer. Therefore, the above hypothesis cannot be taken to have, in itself, a clear constructive meaning.

Instead of establishing a construction for the totality of all models $\mathbf{M}'$ under consideration, we have described, under (7), the construction of a certain semi-model $\mathbf{M}$ which, in a sense, contains a representative selection of models $\mathbf{M}'$. But, besides these models $\mathbf{M}'$, the semi-model $\mathbf{M}$ also contains certain semi-models $\mathbf{M}''$ which are not models or which do not fulfil the conjunctive C.

The semi-model $\mathbf{M}$ can be considered as a finitary spread, each choice sequence in which is a model $\mathbf{M}'$ or at least a semi-model $\mathbf{M}''$. But in general we have no reason to expect that the models $\mathbf{M}'$, in which we are interested, will constitute a subspread $\mathbf{M}^0$ of $\mathbf{M}$.

Now a clear constructive meaning can be connected with quantifiers ranging over *all choice sequences in a given finitary spread* (*cf*. Section 153) and, hence, it makes sense to replace a supposition concerning *all models* $\mathbf{M}'$, which has no clear constructive meaning, by a supposition concerning *all semi-models* $\mathbf{M}''$ (including the models $\mathbf{M}'$). In fact, the above intuitionistic proof depends essentially on a suitable re-interpretation of the above-mentioned hypothesis. It needs hardly saying that this re-interpretation implies a considerable strengthening of the hypothesis.

Example 1. Show that the sequent:

$$(x)A(x) \vee (y)B(y) \vdash (x)(y)[A(x) \vee B(y)]$$

holds true intuitionistically.

Example 2. Show that the sentences:

$$p \to \bar{\bar{p}} \qquad \text{and} \qquad \overline{\overline{(x)[A(x) \vee p] \to \{(x)A(x) \vee p\}}}$$

are intuitionistic theses. — A sentence X is called an *intuitionistic thesis*, if the sequent $\emptyset \vdash X$ holds true intuitionistically.

Example 3. Show that the sentences:

$$p \vee \bar{p},\ \bar{\bar{p}} \to p, \quad \text{and} \quad \overline{\overline{(x)[(y)\{A(x) \vee B(y)\} \to \{A(x) \vee (y)B(y)\}]}}$$

are not intuitionistic theses; construct suitable counter-models.

Example 4. Show that Gentzen's *Hauptsatz* and *Teilformelnsatz* carry over to intuitionistic logic.

Example 5. Discuss the notions of *proof from assumptions* and of a *deductive system* from the point of view of intuitionistic logic. Specifically, find out what part of the results in Section 74 carries over if classical logic is replaced by intuitionistic logic. — Hint: As axioms (I)–(VI) under (T 1) in Section 73 are intuitionistically valid, the same applies to the theses under (XI)–(XVI) in Example 6. Thus the results under (1)–(10) in Section 74 carry over. Go more carefully into the validity of part of the remaining results in that Section.

Example 6. Construct an intuitionistic counterpart to the System N in Section 92.

Example 7. Complete the discussion under (6) by describing the construction steps corresponding to the rules

$$(\mathrm{ij}^{b}),\ (\mathrm{iij}^{a,\,b}),\ (\mathrm{iv}^{a,\,b}),\ (\mathrm{v}^{b}),\ (\mathrm{vi}^{a,\,b}),\ \text{and}\ (\mathrm{vij}^{a,\,b}).$$

Example 8. We say that the construction of semantic tableaux is *normalised*, if a rule is stated which, at each stage in the construction, uniquely determines the next step to be carried out. Establish such a normalisation, taking into account the remark in Section 92, *sub* (2).

Example 9. Complete the proof of the Lemma under (8) by treating the cases

$$(\mathrm{ij}^{b}),\ (\mathrm{iij}^{a,\,b}),\ (\mathrm{iv}^{a}),\ (\mathrm{v}^{a,\,b}),\ (\mathrm{vi}^{a,\,b}),\ \text{and}\ (\mathrm{vij}^{a}).$$

Example 10. Give a more detailed discussion of the introduction of the function c in the above proof of our completeness theorem.

Example 11. Give a proof of our completeness theorem which is based on the Fundamental Theorem on Finitary Spreads.

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