Lecture 22

CS 4860

November 10, 2016

1 Topics

1. Review notion of existence, $\exists x.P(x)$, and disjunction, $A \vee B$, as understood in constructive logic, in contrast with classical logic. In classical logic we can avoid $\exists x.P(x)$ as a primitive by using $\sim \forall x. \sim P(x)$. We can define disjunction as $\sim (\sim A \;\&\; \sim B)$.

2. Give a constructive (intuitionistic) version of FOL, call it iFOL, and define Heyting Arithmetic (HA) as a constructive version of Peano Arithmetic (PA). We reference Kleene's account of PA and HA. We include his axioms in the course notes. These theories of arithmetic provide a basis for defining constructive and classical *real numbers*. This in turn gives us a constructive account of (analytic geometry). We will discuss the notion of synthetic (or "coordinate free") geometry as well. This will lead to topic 3.

3. Show how to present HA and PA in a foundational theory with computational semantics. This kind of theory is nowadays called *type theory*. It's origins are in *Principia Mathematica* by Russell and Whitehead. Church simplified the theory. Then de Bruijn and Martin-Löf stream-lined it using ideas from L.E.J Brouwer.

4. Explore constructive Euclidean geometry using type theory. This will be a constructive synthetic geometry - a new area of investigation.

2 Kleene's Axioms for Heyting Arithmetic and Peano Arithmetic

The only non-constructive axiom is 8^0 $\neg\neg A \supset A$. When this is added to HA, the theory is PA. In these notes we use the successor operation on numerical terms instead of a'. Here are the axioms in this notation (Kleene's numbering).

14. $\forall a, b : \mathbb{N}.\ (S(a) = S(b) \Rightarrow a = b)$
15. $\forall a : \mathbb{N}.\ \sim (S(a) = 0)$
16. $\forall a, b, c : \mathbb{N}.\ (a = b \Rightarrow (a = c \Rightarrow b = c))$
17. $\forall a, b : \mathbb{N}.\ (a = b \Rightarrow s(a) = s(b))$
18. $\forall a, b : \mathbb{N}.\ Add(a, o, a)$ (Kleene writes $a + o = a$)
19. $\forall a, b, z : \mathbb{N}.\ Add(a, b, z) \Rightarrow Add(a, s(b), s(z))$
20. $\forall a, b : \mathbb{N}.\ Mult(a, 0, 0)$
21. $\forall a, b, z : \mathbb{N}.\ Mult(a, b, z) \Rightarrow Add(z, a, y) \quad \& \quad Mult(a, s(b), y)$

We can think of t and x as primitive recursive functions.

18,19 give add $(a, 0) = a$
add $a, s(b) = s(add(a, b))$

20,21 give $mult(a, 0) = 0$
$mult(a, s(b)) = add(mult(a, b), a)$

Kleene's Induction Principle is Axiom 13 :

$$A(0) \ \ \& \ \ \forall x : \mathbb{N}.\ (A(x) \Rightarrow A(s(x)) \Rightarrow \forall x : \mathbb{N}.\ A(x).$$

These axioms have a clear and direct computational meaning. This is expressed will in Constructive Type Theory as we will explore.

CS4860 Lecture 22 ③

Kleene Introduction to Metamathematics

GROUP A. Postulates for the predicate calculus.
GROUP A1. Postulates for the propositional calculus.

1a. $A \supset (B \supset A)$.

1b. $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$.

2. $\dfrac{A,\ A \supset B}{B.}$

3. $A \supset (B \supset A \& B)$.

4a. $A \& B \supset A$.

4b. $A \& B \supset B$.

5a. $A \supset A \vee B$.

5b. $B \supset A \vee B$.

6. $(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$.

7. $(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$.

8°. $\neg\neg A \supset A$.

GROUP A2. (Additional) Postulates for the predicate calculus.

9. $\dfrac{C \supset A(x)}{C \supset \forall x A(x).}$

10. $\forall x A(x) \supset A(t)$.

11. $A(t) \supset \exists x A(x)$.

12. $\dfrac{A(x) \supset C}{\exists x A(x) \supset C.}$

GROUP B. (Additional) Postulates for number theory.

13. $A(0) \& \forall x(A(x) \supset A(x')) \supset A(x)$.

14. $a'=b' \supset a=b$.

15. $\neg a'=0$.

16. $a=b \supset (a=c \supset b=c)$.

17. $a=b \supset a'=b'$.

18. $a+0=a$.

19. $a+b'=(a+b)'$.

20. $a \cdot 0=0$.

21. $a \cdot b'=a \cdot b+a$.

(The reason for writing "°" on Postulate 8 will be given in § 23.)

One may verify that 14—21 are formulas; and that 1—13 (or in the case of 2, 9 and 12, the expression(s) above, and the expression below, the line) are formulas, for each choice of the A, B, C, or x, A(x), C, t, subject to the stipulations given at the head of the postulate list.

The class of 'axioms' is defined thus. A formula is an *axiom*, if it has one of the forms 1a, 1b, 3—8, 10, 11, 13 or if it is one of the formulas 14—21.

The relation of 'immediate consequence' is defined thus. A formula is an *immediate consequence* of one or two other formulas, if it has the form shown below the line, while the other(s) have the form(s) shown above the line, in 2, 9 or 12.

This is the basic metamathematical definition corresponding to Postulates 2, 9 and 12, but we shall restate it with additional terminology

1952, reprinted in 57, 59, 62, 64, 67, 71, 74, 80, 88, 91, 96, and 2000.

3 Number Theory, HA and PA, in Type Theory

We will see how to treat number theory in the setting of modern type theory. Type theories are implemented in proof assistants such as Agda, Coq, Nuprl, and others.

The natural numbers form a type. We denote it $\mathbb{N}$. Types have elements, among them are the *canonical elements*, those that can't be further reduced or "simplified" by computation rules. In Kleene's theory, the *canonical numbers* are

$$0, s(0), s(s(0)), s(s(s(0))), \ldots$$

A non-canonical numerical expression is given by the recursive definitions, e.s. $add(s(s(0)), s(0))$. These reduce by computation rules to canonical numbers, e.s. $s(s(s(0)))$.

In practice, we use an account of numbers based on decimal numbers, $0, 1, 2, \ldots, 9, 10, 11, \ldots$. Non-canonical: 10×11 reducing to 110, $9 \times 9 \times 12$ reducing to 972, 17×17 to 289, etc.