Lecture 21

CS 4860

November 8, 2016

Impact of the Great Schism on Logic and the Foundations of Mathematics.

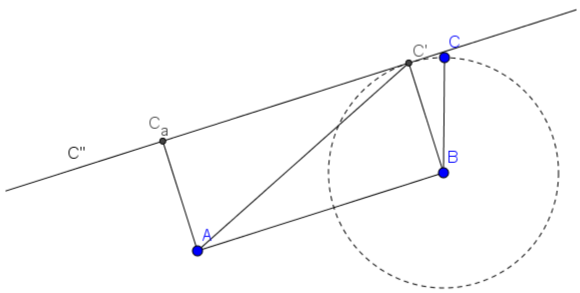
1 Topics

1. Further remarks on Euclidean geometry - discussing Proposition 2.
2. Historical perspective on the schism circa 1880's discussed by Edwards and mentioned in Lecture 19. This split resulted in *constructive* and *intuitionistic* mathematics. These in turn have influenced computer science development, and that has fed back into logical foundations in several ways. Seemed to be resolved in 1900, when Poincaré, in a lecture at International Congress, boasted, "Today there remains in analysis only integers and finite and infinite systems of integers interrelated by a set of relations of equality or inequality."
3. Introducing constructive mathematics in the realm of numbers, e.g. *Peano Arithmetic* and its constructive counter part, *Heyting Arithmetic*.
4. Influence of Bouwer on logic, circa 1907 and beyond.
   - the meaning of existence, $\exists x, t : T.P(x)$.
   - the meaning of $\&, \Rightarrow, \forall$
   - the meaning of $\vee, \exists$
   - issues with real numbers
5. Martin-Löf on numbers as a model for computational reasoning (he was a Kolmogorov student).

2 Remarks on Euclidean Geometry

1. Proposition 2

   We posted Euclid Proposition 2 and the alternative constructive account based on geometry in type theory. Is there a cleaner "Euclid style" proof for Prop 2? How about this:

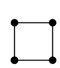
   (a) Connect B to A
   (b) Construct a circle centered at B, radius BC, call it CC_1C_2
   (c) Construct $\perp$ to AB at B, intersect $\bigcirc CC_1C_2$ at C'
   (d) Connect C' to A, forming a right triangle $BC'A$
   (e) Construct line $C'C''$ parallel to AB.
   (f) Drop $\perp$ from A to $C'C''$, at C_a
   (g) AC_a is the required line.

   What must we check to know this proof makes sense? (That we don't need Prop 2 on the constructions we list.)

2. Was Aristotle right that numbers must come before geometry? Can we create "geometric numbers?"

   - 0 is a point
   - 1 is a segment
   - 2 is 2 segments
   - 3 is 3 segments

   From this we can get "square numbers"

3. There remain open problems in Euclidean geometry:
   - Steiner-Lehmus Theorem, posed in 1840:
     Every triangle with equal length angle bisectors is isosceles.
   - Journal of open problems in Euclidean geometry:
     Journal of Classical Geometry (jcgeometry.org).

3 Historical Perspective on the "Great Schism" discussed by H. Edwards

Elements	300 BCE	until today
Ptolemic Period	323 BCE	
	290 BCE	Museum & Library at Alexandria
	125 BCE	Trigonometry
	250 AD	Diophantus
	(200-1200 AD)	Hindu mathematics
	(400-1100 AD)	Arabic mathematics, stagnation in Europe
	1400-1600	Renaissance
		Bacon, da Vince, Copernicus, Kepler
		Descartes, *La Geometriè*
	1601	Galileo 1610
		Fermat 1660
		Descartes 1637, *Discourse on Method*
		Newton 1671, *Principia*
		Leibniz (1646-1716), *De Arte Combinatoria*
	1700	the transformation of mathematics
		the function concept (Euler)
	1800	Installation of rigor
		Gauss,Cauchy,Bolzano, Weierstrauss
	1880	Foundations
		Frege 1879
		Peano 1889
ZFC, Type Theory, Intuitionism	1900	Poincare's address
		Brouwer 1907
		Zermelo 1908
		Fraenkel 1922
		Russell & Whitehead, *Principia Mathematica* 1925
		Hilbert, *On the Infinite*

4 Introducing Constructive Mathematics

Peano Arithmetic - as a model classical system to axiomatize the natural numbers
- uses primitive recursion/induction

5 Influence of L.E.J. Brouwer on Mathematics and Logic

- the meaning of existence $\exists x : \mathbb{N}.P(x)$, $\exists x : \mathbb{R}.P(x)$.
- standard example

  $\exists a, b : \mathbb{R}.$ (irrational (a) & irrational (b) & a^b rational).

A little non-constructive proof:

Consider $\sqrt{2}^{\sqrt{2}}$, is it rational?

yes, then $a = \sqrt{2}$, $b = \sqrt{2}$

no, then $\sqrt{2}^{\sqrt{2}}$ and $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$

Next lecture, 22

- meaning of $\&, \Rightarrow, \forall$
- meaning of $\vee, \exists$
- Law of Excluded Middle $(P \vee \sim P)$

  Brouwer: $\sim\sim (P \vee \sim P)$ in simple words
- issues with real numbers, $\mathbb{R}$
  1. Cannot be made up of smaller bits, say rationals or sequences of rationals.

  We postpone this topic until we have mastered $\mathbb{N}$ from a constructive perspective.