1 Topics

(1) Finishing up non-standard analysis from H. Jerome Keisler’s book *Elementary Calculus* (logician’s pun on “elementary,” it also means “first-order” in some contexts).

(2) Introduction to the classical versus constructive mathematics “divide.” This will take us through

- constructive geometry,
- constructive/computational number theory (Heyting Arithmetic),
- constructive analysis (Brouwer and Bishop), and
- constructive FOL, sometimes called iFOL.

(3) These topics will inform constructive type theory.
Def: $f$ is continuous at a point $c \in \mathbb{R}$ iff

(i) $f$ is defined at $c$ and

(ii) whenever $x$ is infinitely close to $c$ ($x \approx c$),
    $f(x)$ is infinitely close to $f(c)$, i.e. $f(x) \approx f(c)$.

Thm 1: $f$ is continuous at $c$ iff $\lim_{x \to c} f(x) = f(c)$.

Def: $L$ is the limit of $f(x)$ as $x$ approaches $c$ if whenever
$x$ is infinitely close to $c$ but not equal to it, $f(x)$
is infinitely close to $L$, i.e. whenever $x \approx c$ and
$x \neq c$, $f(x) \approx L$.

Note: We can have $\lim_{x \to c} f(x)$ yet $f(c)$ is not defined.

Def: (p.73) Given $f \in \mathbb{R} \to \mathbb{R}$, the derivative of $f$ is the function $f'$ whose value at $x$ is the slope of $f$ at $x$,
i.e. $f'(x) = \text{st} \left( \frac{f(x + \Delta x) - f(x)}{\Delta x} \right)$ whenever the slope exists.
3 Constructive vs Classical Mathematics

(from Harold M. Edwards: Essays in Constructive Mathematics)

Preface

Mathematics came to a fork in the road around 1880

Dedekind, Cantor, Weierstrass vs Kronecker, (spirit of Gauss, Dirichlet)
(accepted “transfinite functions”) later Brouwer, Poincaré, Bishop

“The advent of computers has had a profound impact on mathematics, directing attention to algorithms.”

We are not justified in this course to pursue the theory of polynomials that Edwards uses to illustrate constructivity. Moreover, there are good examples closer to Computer Science (CS) that make his points.

We can rely on a simpler and more universal set of concepts arising from Euclidean Geometry, a theory with constructive aspirations in which computations are essential.

We will look at the historical progression of constructive mathematics.

I. Euclidean Geometry - naive and informal, synthetic geometry

Euclid’s Elements 300 BC Alexandria, Proclus commentary

Appollonius Died in 190 BC

Book VIII Euclid Proposition 20: There are infinitely many primes (prime numbers are more than any assigned magnitude of prime numbers).

II. Descartes - analytic geometry

Discourse on Method 1637, with appendices

La Géométrie Coordinate geometry (only book he wrote on mathematics)

Rules for the Direction of Mind Isolated from Discourse

(a) I think, therefore I am
(b) Each phenomenon must have a cause
(c) An effect cannot be greater than its cause
(d) The mind has innate (knowledge) of perfection, space, time, motion
   (The idea of perfection cannot be created by the imperfect mind of man, thus God exists. God made nature to follow mathematical laws.)
III. Dedekind - (1888) Recursive definitions

IV. Peano - (1889) The principles of arithmetic presented by a new method

(He used recursive definitions and induction. Next we look at primitive recursive definitions of numerical functions.)

4 Examples of Primitive Recursion

\[ a_0(x, y) = x + 1 \]
\[ s(x) = x + 1 \quad \text{so} \quad a_0 = s(x) \]

\[ a_1(x, y) = \text{add}(x, y) \]
\[ \text{add}(0, y) = y \]
\[ \text{add}(s(x), y) = s(\text{add}(x, y)) \]

note \[ \text{add}(x, y) = a_0(a_1(x, y), y) \]

\[ a_2 = \text{mult}(x, y) \]
\[ \text{mult}(0, y) = 0 \]
\[ \text{mult}(s(x), y) = \text{add}(\text{mult}(x, y), y) \]

i.e. \[ (x + 1) \cdot y = x \cdot y + y \]

note \[ \text{add}(\text{mult}(x, y), y) = a_1(a_2(x, y), y) \]

\[ a_3(x, y) = \text{exp}(x, y) \]
\[ \text{exp}(0, y) = 1 \quad \text{i.e.} \quad y^0 = 1 \]
\[ \text{exp}(s(x), y) = \text{mult}(\text{exp}(x, y), y) \]

i.e. \[ y^{x+1} = y^x \cdot y \]

note \[ \text{mult}(\text{exp}(x, y), y) = a_2(a_3(x, y), y) \]

\[ a_4(x, y) = \text{hypexp}(x, y) \]
\[ \text{hypexp}(0, y) = y \]
\[ \text{hypexp}(s(x), y) = \text{exp}(\text{hypexp}(x, y), y) \]

note \[ \text{exp}(\text{hypexp}(x, y), y) = a_3(a_4(x, y), y) \]

\[ a_{n+1}(x, y) \]
\[ a_{n+1}(0, y) = y \]
\[ a_{n+1}(s(x), y) = a_n(a_{n+1}(x, y), y) \]

\[ a_n(x, y) \text{ can be thought of as a function of } n, x, y. \text{ This is Ackerman's function in one form. It is not primitive recursive.} \]