Lecture 15

CS 4860

October 2016

The topic of this lecture is Smullyan's proof of the completeness of the Tableau Axioms for First-Order Logic. This is covered on pages 57-61, just 5 pages of clever and elegant logic and the core of his book *First-Order Logic* from 1968. Please study these pages of Chapter 5. We will discuss the ideas in lecture. Getting this result for a programming logic, say based on the refinement method, is a new result not yet discussed in *any textbook*. We will see why it is not typically covered and how the situation might change in a significant and practical way.

Smullyan starts on p.57 saying "Now we turn to the proof of the major results in quantification theory: Every valid sentence (of FOL) is provable by the tableau method," my parenthetical comment. Also, I would say "we turn to a proof." I would also stress that it is the most elegant and accessible proof I know.

In 1928 Hilbert posed three open problems at an international mathematics conference. The third became known as the *Entscheidungsproblem* [1] (discussed even by Leibniz in an informal way earlier).

A related result we will not cover is *Church's Theorem* that there is no procedure to decide whether an arbitrary FOL proposition is provable. Church made the *Entscheidungsproblem* precise, defining computability, and solved it in 1936. Turing independently solved it soon after.

Def. *Hintikka Sets* for Universe U a set of U-formulas such S such that for every α, β, γ, and δ we know: p. 57 bottom

- $\mathbf{H_0}$: No atomic element (prop) of E^u and its negation belong to S. (Recall E^u are the closed U-formulas-no free varaibles.)

- $\mathbf{H_1}$: If $\alpha \in S$ then α_1, x_2 are in S. (Recall α formulas are *signed formulas* $T(X\&Y)$, $F(X \vee Y)$, $F(X \Rightarrow Y), T \sim X, F \sim X$.)

- $\mathbf{H_2}$: If $\beta \in S$, then $\beta_1 \in S$ or $\beta_2 \in S$. (Recall, β are $F(X\&Y), T(X \vee Y), T(X \Rightarrow Y)$. See p.21.)

- $\mathbf{H_3}$: If $\gamma \in S$ then for every parameter $k \in U, \gamma(k) \in S$. (Recall γ are $T(\forall x.A(x))$ of $F(\exists x A(x)$.)

- $\mathbf{H_4}$: If $\delta \in S$ then for at least one element $k \in U$, $\delta(k) \in S$. (Recall δ are $T\exists x A(x)$, $F(\forall x.A(x))$ and we need a provision for $A(k)$ and $\sim A(k)$ that k is new.)

[1]It is a common misunderstanding that Hilbert posed this open problem in his list of 23 problems presented at the 1900 International Congress of Mathematicians.

Hintikka's Lemma: Every Hintakka set S for a domain U is satisfiable in U. (p.58)

Proof: We need to find an *atomic valuation* in which all elements of the Hintikka set are true.

(1) We define an *atomic valuation*, degree 0: For ever atomic sentence $P\ c_1, \ldots, c_n$ (p.43) Smullyan uses $P\ \epsilon_1, \ldots, \epsilon_n$ where ϵ_i are variables or elements of U but not parameters (p.47).

(2) Next, we pick any X of positive degree and assume the result for all elements of lower degree. We go by cases $\alpha, \beta, \gamma, \delta$. The new cases are γ and δ.

How to use Hintakka's Lemma for completeness?

The interesting case is when the Tableau procedure can continue indefinitely.

How to be sure that any open ∞ branch will give us a Hintakka set?

The new cases beyond propositional logic are only γ, δ. (p.58)

- γ case: for every $k \in U, \gamma(k)$ is of lower degree, hence true by induction hyp.
- δ case: for at least one $k \in U$, $\delta(k) \in S$ (by H_4, hence $\delta(k)$ is true by induction hyp.)

Qed

Note, the proof procedure might not close because it runs "forever." This would generate an ∞ path, (König's Lemma). Many such paths are not Hintakka sets.

How to guarantee a Hintakka set? See Smullyan, top of p.59