1 Smullyan’s Proof Rules page 54

We need the 8 rules of Propositional Logic - page 17, 20, 21.
We also use the $\alpha$ and $\beta$ classification.

<table>
<thead>
<tr>
<th>Rule A</th>
<th>$\alpha$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$\alpha_1$</td>
<td>$\alpha_2$</td>
<td></td>
</tr>
<tr>
<td>$T(X \land Y)$</td>
<td>$TX$</td>
<td>$TY$</td>
<td></td>
</tr>
<tr>
<td>$F(X \lor Y)$</td>
<td>$FX$</td>
<td>$FY$</td>
<td></td>
</tr>
<tr>
<td>$F(X \Rightarrow Y)$</td>
<td>$TX$</td>
<td>$FY$</td>
<td></td>
</tr>
<tr>
<td>$T(\neg X)$</td>
<td>$FX$</td>
<td>$FX$</td>
<td></td>
</tr>
<tr>
<td>$F(\neg Y)$</td>
<td>$TX$</td>
<td>$TX$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Rule B</th>
<th>$\beta$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>$\beta_1$</td>
<td>$\beta_2$</td>
<td></td>
</tr>
<tr>
<td>$F(X \land Y)$</td>
<td>$FX$</td>
<td>$FY$</td>
<td></td>
</tr>
<tr>
<td>$T(X \lor Y)$</td>
<td>$TX$</td>
<td>$TY$</td>
<td></td>
</tr>
<tr>
<td>$T(X \Rightarrow Y)$</td>
<td>$FX$</td>
<td>$TY$</td>
<td></td>
</tr>
</tbody>
</table>
We add Rule C and Rule D - page 53

**Rule C**

\[
\frac{\gamma}{\gamma(a)}
\]

\(\gamma\) are *universal* type; \(\forall x.A, \sim \exists x.A\)

\[
\frac{T(\forall x.A)}{T A(a/x)} \quad \frac{F(\exists x.A)}{F A(a/x)}
\]

any parameter \(a\)

**Rule D**

\[
\frac{\delta}{\delta(a)}
\]

\(\delta\) are *existential* type; \(\exists x.A, \sim \forall x.\)

\[
\frac{T(\exists x.A)}{T A(a/x)}
\]

\(\text{new } a\)

\[
\frac{F(\forall x.A)}{F A(a/x)}
\]

\(\text{new } a\)

also see liberalized rule D - either \(a\) is never or has not been previously introduced by rule D. An example is given below in Proof 2.
2 Sample Proofs

\[ \exists y (P(y) \Rightarrow \forall x. P(x)) \]

Example 2

Proof 1
p.56

(1) \[ F (\exists y (\exists x. P(x) \Rightarrow P(y))) \] Strict rule D

(2) \[ F \exists x. P(x) \Rightarrow P(a) \] \( a \) introduced by rule D from (1)

(3) \[ T \exists x. P(x) \] \( F(X \Rightarrow Y) \) rule applied at (2) to give (3), (4) an \( \alpha \)-rule

(4) \[ F P \ a \]

(5) \[ T P \ b \] Line (5) is rule D on (3) new \( b \)

(6) \[ F (\exists x. P(x) \Rightarrow P(b)) \] Strict rule D from (1)

(7) \[ F P \ b \]
\[ \# \ 5 \ & \ 7 \]

Proof 2
p.56

(1) \[ F (\exists y. (\exists x. P(x) \Rightarrow P(y))) \] Rule D, use \( a \) just as in Proof 1

(2) \[ F \exists x. P(x) \Rightarrow P(a) \] Conclusion of Rule D with \( a \) for \( y \)

(3) \[ T \exists x. P(x) \]

(4) \[ F P \ a \]

(5) \[ T P \ a \]
\[ \# \]
\[ \# \ 5 \ & \ 7 \]

Liberal rule D with \( a \) from (3), allowed since the \( P(a) \) in \( F \ P(a) \) does not come from rule D, it is from \( F(X \Rightarrow Y) \) in line 2.
Challenge Example - Give justifications

(1) F \( (\exists y (P(y) \Rightarrow \forall x. P(x))) \)

(2) F \( (P(a) \Rightarrow \forall x. P(x)) \)

(3) T \( P(a) \)

(4) F \( \forall x. P(x) \)

(5) F \( P(a) \)  

Rule D
3 Extended Proofs for Example 2

Proofs 1 and 2 of Example 2 page 55 bottom and 56 top, illustrating strict rule D:

(1) $F \left( \exists y (\exists x. P(x) \Rightarrow P(y)) \right)$

use strict rule D with new $a$.

Set the goal to attempt to falsify the formula. We can always assume that the domain of discourse is inhabited with at least one element. This proposition assumes a value for $\exists y$. We let $a \in D$ be that value and express this in step 2.

(2) $F \exists x. P(x) \Rightarrow P(a)$

To falsify (2) we use the propositional rule for $F(X \Rightarrow Y)$ from page 17. The conclusions are lines (3) and (4).

(3) $T \exists x. P(x)$

from (2) by $F (X \Rightarrow Y)$

(4) $F P a$

from (2) by $F (X \Rightarrow Y)$

(5) $T P b$

Now from line (3) $T \exists x. P(x)$, we use rule D $T \frac{\exists x. P(x)}{T P(b)}$.

(6) $F (\exists x. P(x) \Rightarrow P b)$

from (1) by strict rule D (note, the same form as (2) but with $b$ new).

(7) $F P b$

# 5 & 7

(7) from (6) $F(\exists x P(x) \Rightarrow P(b))$ by $F(X \Rightarrow Y)$ but we don’t need to repeat $T \exists x. P(x)$, so we only show $F P(b)$ as line 7.

We can get a shorter proof by noting the $F P(a)$ in line 4 comes from $F (X \Rightarrow Y)$ not from rule D. We show that shorter proof below.
Proof 2

(1) F (∃y.(∃x.P(x) ⇒ P(y)))

Rule D, use a just as in Proof 1 apply rule C with a to get (2).

(2) F ∃x.P(x) ⇒ P(a)

Use F(X ⇒ Y) on (2).

(3) T ∃x.P(x)

from (2) by F (X ⇒ Y)

(4) F P a

from (2) by F (X ⇒ Y)

(5) T P a

# from (3) since a was not introduced by Rule D in line 4, we can use a in rule D from 3.

**Note:**
There is an important lesson from Lect. 13 that I will mention now in retrospect. It is important for a proof to give clear justifications of every step because it is not clear from the sequence or tree of formulas.
(0) $F \left( \forall x. (Px \Rightarrow Qx) \right) \Rightarrow (\forall x P(x) \Rightarrow \forall x Q(x))$

(1) $T \left( \forall x. (P(x) \Rightarrow R(x)) \right)$

(2) $F \left( \forall x. P(x) \Rightarrow \forall x Q(x) \right)$

(3) $T \forall x. P(x)$

(4) $F \forall x. Q(x)$

(5) $F Q(a)$ new $a$

(6) $T P(a)$

(7) $T \left( P(a) \Rightarrow Q(a) \right)$

(8) $F P(a) \quad \mid T Q(a)$
   # with 6 \quad \# with 5
4 Summarize Smullyan Completeness Argument and Consistency

Consistency p.53, proof page 55, one paragraph:

- $G_1$: If $S$ is satisfiable and $\alpha \in S$, then $\{S, \alpha_1, \alpha_2\}$ is satisfiable.
- $G_2$: If $S$ is satisfiable and $\beta \in S$, then at least one of $\{S, \beta_1\}, \{S, \beta_2\}$ is satisfiable.
- $G_3$: If $S$ is satisfiable and $\gamma \in S$, then for every parameter $a$, $\{S, \gamma(a)\}$ is satisfiable.
- $G_4$: If $S$ is satisfiable and $\delta \in S$, and $a$ is a parameter not occurring in $S$, then $\{S, \delta(a)\}$ is satisfiable.

Consistency

| Lemma p.55 | Any immediate extension of a tableaux with at least one open branch, i.e. satisfiable branch, remains satisfiable. |
| Theorem p.55 | Thus if a tableau closes, then the goal is unsatisfiable, so $G$ is valid. This it the same idea (plan) as for propositional logic consistency, provable implies valid. |

Completeness

| Theorem 3 p.60 | Theorem of completeness. |
| Theorem Löwenheim p.61 | If $X$ is satisfiable, then it is satisfiable in a denumerable domain. |