Lecture 13

CS 4860

October 6, 2016

1 Smullyan's Proof Rules page 54

We need the 8 rules of Propositional Logic - page 17, 20, 21.

We also use the α and β classification.

Rule A

$$\frac{\alpha}{\begin{array}{c}\alpha_1 \\ \alpha_2\end{array}}$$

α are *conjunctive* type

α	α_1	α_2
$T(X \wedge Y)$	TX	TY
$F(X \vee Y)$	FX	FY
$F(X \Rightarrow Y)$	TX	FY
$T(\sim X)$	FX	FX
$F(\sim Y)$	TX	TX

Rule B

$$\frac{\beta}{\beta_1 \mid \beta_2}$$

β are *disjunctive* type

β	β_1	β_2
$F(X \wedge Y)$	FX	FY
$T(X \vee Y)$	TX	TY
$T(X \Rightarrow Y)$	FX	TY

We add Rule C and Rule D - page 53

Rule C

$$\frac{\gamma}{\gamma(a)}$$

γ are *universal* type; $\forall x.$A, $\sim \exists x.$A

$$\frac{T(\forall x.\mathrm{A})}{T\ A(a/x)} \qquad \frac{F(\exists x.\mathrm{A})}{F\ A(a/x)}$$

any parameter a

Rule D

$$\frac{\delta}{\delta(a)}$$

δ are *existential* type; $\exists x.$A, $\sim \forall x.$

$$\frac{T(\exists x.\mathrm{A})}{T\ A(a/x)}$$

new a

$$\frac{F(\forall x.\mathrm{A})}{F\ A(a/x)}$$

new a

also see liberalized rule D - either a is never or has not been previously introduced by rule D. An example is given below in Proof 2.

2 Sample Proofs

$\exists y(P(y) \Rightarrow \forall x.P(x))$

Example 2
Proof 1
p.56

(1) F $(\exists y(\exists x.P(x) \Rightarrow P(y)))$	Strict rule D
(2) F $\exists x.P(x) \Rightarrow P(a)$	a introduced by rule D from (1)
(3) T $\exists x.P(x)$	$F(X \Rightarrow Y)$ rule applied at (2) to give (3), (4) an α-rule
(4) F $P\ a$	
(5) T $P\ b$	Line (5) is rule D on (3) new b
(6) F $(\exists x\ P(x) \Rightarrow\ P\ b)$	Strict rule D from (1)
(7) F $P\ b$ # 5 & 7	

Proof 2
p.56

(1) F $(\exists y.(\exists x.P(x) \Rightarrow P(y)))$	Rule D, use a just as in Proof 1
(2) F $\exists x.P(x) \Rightarrow P(a)$	Conclusion of Rule D with a for y
(3) T $\exists x.P(x)$	
(4) F $P\ a$	
(5) T $P\ a$ #	Liberal rule D with a from (3), allowed since the $P(a)$ in F $P(a)$ does not come from rule D, it is from F($X \Rightarrow$ Y) in line 2.

Challenge Example - Give justifications

(1) F $(\exists y(P(y) \Rightarrow \ \forall x.P(x)))$

(2) F $(P(a) \Rightarrow \forall x.P(x))$

(3) T $P(a)$

(4) F $\forall x.P(x)$

(5) F $P(a)$ Rule D

3 Extended Proofs for Example 2

Proofs 1 and 2 of Example 2 page 55 bottom and 56 top, illustrating strict rule D:

Proof 1

(1) F $(\exists y(\exists x.P(x) \Rightarrow P(y)))$ *use strict rule D with new a.*	Set the goal to attempt to falsify the formula. We can always assume that the domain of discourse is inhabited with at least one element. This proposition assumes a value for $\exists y$. We let $a \in D$ be that value and express this in step 2.
(2) F $\exists x.P(x) \Rightarrow P(a)$	To falsify (2) we use the propositional rule for $F(X \Rightarrow Y)$ from page 17. The conclusions are lines (3) and (4).
(3) T $\exists x.P(x)$	from (2) by $F\ (X \Rightarrow Y)$
(4) F $P\ a$	from (2) by $F\ (X \Rightarrow Y)$
(5) T $P\ b$	Now from line (3) $T\ \exists x.P(x)$, we use rule D $\frac{T\ \exists x.P(x)}{T\ P(b)}$.
(6) F $(\exists x\ P(x) \Rightarrow\ P\ b)$ *from (1) by strict rule D (note, the same form as (2) but with b new).*	(7) from (6) $F(\exists x P(x) \Rightarrow P(b))$ by $F(X \Rightarrow Y)$ but we don't need to repeat $T\ \exists x.P(x)$, so we only show $F\ P(b)$ as line 7.
(7) F $P\ b$ # 5 & 7	

We can get a shorter proof by noting the $F\ P(a)$ in line 4 comes from $F\ (X \Rightarrow Y)$ not from rule D. We show that shorter proof below.

Proof 2

(1) F $(\exists y.(\exists x.P(x) \Rightarrow P(y)))$	Rule D, use a just as in Proof 1apply rule C with a to get (2).
(2) F $\exists x.P(x) \Rightarrow P(a)$	Use $F(X \Rightarrow Y)$ on (2).
(3) T $\exists x.P(x)$	from (2) by $F\ (X \Rightarrow Y)$
(4) F $P\ a$	from (2) by $F\ (X \Rightarrow Y)$
(5) T $P\ a$ #	from (3) since a was not introduced by Rule D in line 4, we can use a in rule D from 3.

$**$Note:
There is an important lesson from Lect. 13 that I will mention now in retrospect. It is important for a proof to give clear justifications of every step because it is not clear from the sequence or tree of formulas.

Page 55 using signed formula (the book uses unsigned).

(0) F $(\forall x.(Px \Rightarrow Qx)) \Rightarrow (\forall x P(x) \Rightarrow \forall x.Q(x))$

(1) T $(\forall x.(P(x) \Rightarrow R(x))$

(2) F $(\forall x.P(x) \Rightarrow \forall x.Q(x))$

(3) T $\forall x.P(x)$

(4) F $\forall x.Q(x)$

(5) F $Q(a)$ new a

(6) T $P(a)$

(7) T $(P(a) \Rightarrow Q(a))$

(8) F $P(a)$ | T $Q(a)$
with 6 | # with 5

4 Summarize Smullyan Completeness Argument and Consistency

Consistency p.53, proof page 55, one paragraph:

- G_1: If S is satisfiable and $\alpha \in S$, then $\{S, \alpha_1, \alpha_2\}$ is satisfiable.
- G_2: If S is satisfiable and $\beta \in S$, then at least one of $\{S, \beta_1\}$,$\{S, \beta_2\}$ is satisfiable.
- G_3: If S is satisfiable and $\gamma \in S$, then for every parameter a, $\{S, \gamma(a)\}$ is satisfiable.
- G_4: If S is satisfiable and $\delta \in S$, and a is a parameter *not* occurring in S, then $\{S, \delta(a)\}$ is satisfiable.

Consistency	
Lemma p.55	Any immediate extension of a tableaux with at least one open branch, i.e. satisfiable branch, remains satisfiable.
Theorem p.55	Thus if a tableau closes, then the goal is unsatisfiable, *so G is valid.* This it the same idea (plan) as for propositional logic consistency, provable implies valid.

Completeness	
Theorem 3 p.60	Theorem of completeness.
Theorem Löwenheim p.61	If X is satisfiable, then it is satisfiable in a denumerable domain.