Lecture 12

CS 4860

October 4, 2016

1 Plan

We want to develop a theory of polymorphic dependent types. An example would be a dependent pair $(x : \alpha * \beta(x))$. For example, we might want to define rational numbers as $int * int$, giving the numerator and denominator as a pair (3,5), (1,4), etc. But we don't want (1,0) or any (n,0). We'd like to write a type such as $int * (x : int \text{ where } x \neq 0)$. This is a dependent type.

The logic that uses ideas like this is FOL. It is the base logic for mathematics in which we define Peano Arithmetic (PA), real analysis, and ZFC set theory.

I think it would be best for us to cover Smullyan because his account is so elegant and thorough. It will lead us to our first deep result, Gödel's Completeness Theorem for FOL, his PhD thesis, 1930, and article "The Completeness of the Axioms of the Functional Calculus of Logic" (in Heijenoort, J. V. (1967). *From Frege to Gödel; a source book in mathematical logic*, 1879-1931. Cambridge: Harvard University Press, 583-591).

The challenge problem for today is an exercise from Smullyan, page 56:

$$\exists y[P(y) \Rightarrow \forall x.P(x)].$$

How can this be true? We have the standard assumption for FOL that the domain of discourse, D, is non-empty. So we can assume a parameter d_0 in D.

2 Some History of FOL

The origin of FOL is quite clear, Gottlob Frege conceived it and wrote "the book on it" in 1879, entitled *Begriffsschrift* (idea writing), a formula language, modeled upon that of arithmetic for pure thought. It is "perhaps the most important single work ever written in logic." Frege was struggling to understand the notion of sequence in mathematics (we are still struggling with this idea, but Frege gave us one great tool). He was inspired by Leibniz and his quest for a "lingua characteristica."

"In particular, I believe that the replacement of the concepts of *subject* and *predicate* by argument and function, respectively, will stand the test of time."(From 1879, still going stron in 2016, 137 years later!)

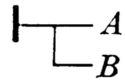

Figure 1: $A \Rightarrow B$. | is the condition stroke.

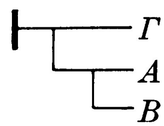

Figure 2: If A is a consequence of B then we can infer Γ.

this stands for the judgment that, whatever we may take for its argument, the function is a fact.

Figure 3: For all a, $\Phi(a)$.

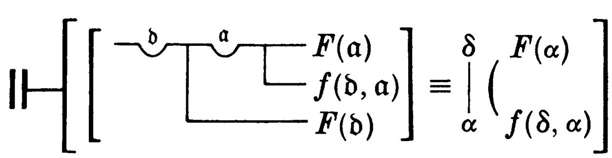

Figure 4: A theorem from the general theory of sequents.

First-Order Logic is the central formal theory in virtually all logic courses. It represents a major discovery in mathematical logic, due in large part to G. Frege in 1879 in his article *Begriffsschrift*.

We are interested in this logic for another key reason, it provides an example for how we can understand and use *dependent types in computing*.

Recall our definition of dependent types from Lecture 11. To define these types, we need a concept like that of a predicate. It is also common to call $P(x_1, \cdots, x_n)$ or $P(a_1, \cdots, a_n)$ predicates. The x_i are *variables* and the a_i are called *parameters* by Smullyan. What is the difference? Smullyan also calls these *atomic* formulas. Why is that a good name? Here are the topics you should study from Smullyan.

1. Formulas (p.43), a recursive definition
2. Substitution (p.44)
3. Closed formulas (p.44)
    - bound variables
    - open and closed formulas
4. Subformulas (p.46)
5. Models of FOL
    - universe of individuals - U : Smullyan's terminology.
    - domain of discourse - D : Refinement Logic terminology.
6. Substitutions (state) $\mathcal{S}(exp) \in D$
    - U-formulas (p.47 top)
    - E^U; all closed U-formulas